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## 1 Introduction

Network structures have been used in many economic situations to describe and analyze interactions between players. The role played by the network relationships in determining the outcome of interactions is now well documented and recognized. The realization of the important role played by network relationships together with the impact of game-theoretic reasoning in economic models has resulted in the birth of a literature that uses game-theoretic reasoning to develop models of network relationships. In this paper, we adopt a particular modeling approach that consists in analyzing networks from a cooperative point of view. In our framework players may form bilateral links. These links capture the way players can collaborate to achieve some productive value. A network is thus a particular architecture representing the pattern of cooperation between the players, and the value generated depends on the particular architecture of cooperation. Players are free to sever any link in which they are involved but the consent of two players is needed to build a link between them. Our goal is to extend and adapt various methods and concepts of cooperative coalitional game theory to network settings. This extension provides a nice basis for making prediction about the outcome of bargaining among players involved in a network relationship and also for the analysis of power or influence of the players in a given network. The place of a player in a network affects her bargaining position and own productivity but also the bargaining positions and productivities of the other players in the network. Since the place of a player in the network depends on her links and the overall structure of the network, we conduct our analysis on a *link-basis*, i.e., the bargaining position and productivity of a player in the network is deduced from the bargaining positions and productivities

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of the links she is involved in.

This paper exploits the novel approach that consists of using the value function of a network game as the characteristic function of a corresponding cooperative game (see definitions infra). Slikker and van den Nouweland [19] refer to this cooperative game as the *link game* associated with the value function of a network game.[1]

In a seminal paper, Myerson [12] leads the way in adapting cooperative game theory structure to take into consideration information about the network connecting players. cooperative coalitional games augmented by a network are called *communication games*. One can view communication games as specific forms of cooperative games where the role of the network is to determine which coalitions can function in the coalitional game, so that each network and characteristic function (determining the value of each possible coalition) induce a particular cooperative game. This induced game is called the *network-restricted game* in Slikker et al. [19]. Myerson proposes then a method to allocate the value of the grand coalition : the Myerson value, which is the Shapley value of the network-restricted cooperative game. Jackson and Wolinsky [9] go further and consider a setting in which economic possibilities depend directly on the network structures connecting players, whereas in network-restricted games the primitives are how the players might be grouped into coalitions. They call this setting *network games*. Jackson and Wolinsky [9] study the relationship between the set of networks that are productively efficient and the set of networks that are (pairwise) stable. They also propose a generalization of the Myerson value in the context of network games. In another paper, Jackson [7] suggests that network games can be thought of as the analog of cooperative games. The motivation of this paper is to draw benefit from this analogy to adapt and extend tools of cooperative games into the context of network games. Indeed, we assume that allocations are link-based, that is, any payoff distribution is realized through the links. The idea of assigning values to links first and then to players rather than directly to players was originated by Meessen [11] and Borm *et al.* [3]. It results in a variation of the Myerson value called the *Position value* for communication games. In our paper, the characterization we propose for the Shapley value as link-based solution for network games, which is the posi-

[1] Despite various authors in the literature refer to the same objects, there is no harmony in the vocabulary they use. In this note, we use the vocabulary of Jackson [7].

tion value for network games, follows closely the approach undertaken by Borm *et al.* [3]. In particular, their characterization is only valid for coalitional games restricted by undirected trees. We show that the transposition of network games into TU games renders the domain rich enough so that the translation of the original Shapley's axioms (closely related to the axioms proposed by Borm *et al.*) are necessary and sufficient to characterize the position value. Note that the position value for general communication games, i.e. graph-restricted coalitional games with any kind of graphs, has been characterized by Slikker [17] and in Slikker [18] as the marginal value of a unique link-potential function.

In the next section we provide the definitions of TU cooperative games and network games, as well as the link game associated with a value function of a network game. In the third section, we adapt the cooperative game notions of core and allocations to network games. We also characterize the set of network games with non-empty cores. Section 4 is devoted to simple network games and their cores. On the fifth section, we present and characterize the (position) Shapley value as a link-based solution concept for network games. In section 6 we provide an example comparing classic allocation rules to the link-based Shapley value. Last section concludes.

## 2 Cooperative games and network games

### 2.1 TU cooperative games

A *cooperative game with transferable utility* (TU game) is a pair $(N, w)$ where $N = \{1, \ldots, n\}$ is the set of players and $w$ is the *characteristic function* of the game, which assigns some real value to any subset $S$ of $N$. We assume that $w(\emptyset) = 0$. Because each $w$ has the family $2^N$ of all possible subsets of $N$ as domain, each $w$ can be seen as a vector in $\mathbb{R}^{2^n-1}$. In a game $(N, w)$, a subset $S$ of $N$ is called a *coalition* and $w(S)$ is the *worth* the coalition $S$ can guarantee to itself without the help of the other players in $N \setminus S$. The set of all possible games on a player set $N$ is denoted $W(N)$. In most cases we treat the player set as fixed and $w$ alone designates the *game*. The term "transferable utility" conveys the idea that the worth generated by a coalition can be freely transferred among its members.

A *basis* for the set of games $W(N)$ consists of a list of games $w_k$, $k = 1 \ldots 2^n - 1$

such that any game in $W(N)$ can be written as a linear combination of the games in the basis and no game in the basis can be written as a linear combination of the remaining games in the basis.

How the value of the player set $N$ is allocated among its member is captured by a *solution* or *value* for TU games. A solution is a function $\theta : W(N) \to \mathbb{R}^N$. A solution is*efficient* if $\sum_i \theta_i(w) = w(N)$. A solution indicates how much of the value of the player set $N$ each player receives. It is thus presumed that the grand coalition of the player set $N$ generates the maximum value.[2] A prominent solution is the Shapley value and it is defined as follows :

$$\theta_i^{Sh}(w) = \sum_{S \subset N \setminus \{i\}} (w(S \cup \{i\}) - w(S)) \left( \frac{|S|!(n-|S|-1)!}{n!} \right) \tag{2.1}$$

where $|S|$ denotes the number of players in coalition $S$ and $n$ the number of players in $N$.

For a throughout treatment of TU (and non TU) cooperative games, the reader is referred to Peleg and Sudhölter [14].

## 2.2 Communication games and Network games

A *network* $g$ lists which pairs of players are linked to each other. A network is thus a list of *unordered* pairs of players $\{i, j\}$ or $\{j, i\}$ where $i \in N$ and $j \in N$, $i \neq j$. For simplicity, we use the notation $ij$ to designate the pair $\{i, j\}$. We sometimes designate links by the letters $l$ or $l'$ without mentioning the label of the players involved in the links. The complete network on $N$, in which any two players are linked is denoted $g^N$ and is the set of all subsets of $N$ of size 2. We denote the set of all possible networks on $N$ by $G(N) \equiv \{g : g \subseteq g^N\}$.

We denote the network obtained by adding a link between $i$ and $j$ by $g + ij$ and we denote the network obtained by deleting the link $ij$ from an existing network $g$ by $g - ij$. The networks $g + l$ and $g - l$ are defined similarly, with $l$ a link between two players.

The set of players who have at least one link in a network $g$ is $N(g) \equiv \{i : \exists j \text{ s.t. } ij \in g\}$ and $n(g)$ is the cardinality of $N(g)$, i.e. the number of players involved in a network relationship at $g$.

[2]To be precise, it is assumed that the value to be distributed among players is the value of the grand coalition, and hence it is assumed that $N$ generates the maximum value.

Let $L_i(g)$ be the set of links of player $i$ in the network $g : L_i(g) \equiv \{ij : ij \in g\}$. If $l_i(g)$ is the cardinality of $L_i(g)$ then $\sum_i \frac{1}{2} l_i(g) \equiv l(g)$ is the total number of links in $g$.

For any given coalition $S \subseteq N$ and network $g$, we denote by $g^S$ the complete network among the players in $S$[3] whereas $g_{|S}$ is the network found by deleting all links of $g$ involving at least one player outside $S$, i.e. $g|_S \equiv \{ij : ij \in g \text{ and } i \in S, j \in S\}$.

A *component* of a network $g$ is a nonempty maximally connected subnetwork $g' \subseteq g$ such that

- if $i \in N(g')$ and $j \in N(g')$, $i \neq j$, then there exists a path in $g'$ between $i$ and $j$, i.e., a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g'$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$, and
- if $i \in N(g')$ and $ij \in g$ then $ij \in g'$.

Thus, the components of a network $g$ partition the player set $N$ into distinct connected subgraphs of a network.[4] The set of components of a network $g$ is denoted $C(g)$.

A *communication game* is a triple $(N, w, g)$ such that $(N, w) \in W(N)$ is a TU game and $g \in G(N)$ a network on the player set $N$. The TU game is augmented by a network that determines who can communicate with whom. Each communication game $(N, w, g)$ induces a TU game $(N, \hat{w}_g)$ such that

$$\hat{w}_g(S) = \sum_{C \in C(g|_S)} w(C)$$

where the sum is over all coalitions $C$ of the partition $C(g_{|S})$ of $S$ generated by the components of $g$ restricted to the players in $S$. The game $(N, \hat{w}_g)$ is called the *network restricted game* associated with the communication game $(N, w, g)$. A coalition in this setting generates some worth to the extent that the members of the coalition can communicate.

One way to adapt the concept of solution in the context of communication games is proposed by Myerson [12]. The *Myerson value* is a natural extension of

[3]By consequence of the definition of $g^S$ for all $S \subseteq g^N$, we have that $g^{\{i\}} = g^{\emptyset}$ the empty network, since it makes at least two players to make a link. Also we have $N(g^{\{i\}}) = N(g^{\emptyset})$.

[4]By the definition of component based on identifying players in $N(g)$, an isolated player (with no link) is not a component since it never belongs to any $N(g)$. Thus, the set of components forms a partition of the player set $N$ if only if there is no isolated player.

the Shapley value (equation (2.1)) to communication games and it is defined by

$$\psi^M(N,w,g) = \theta^{Sh}(N,\hat{w}_g).$$

The Myerson value for a communication game is thus the Shapley value of the induced network restricted TU game. Note that if the network $g$ is the complete network $g^N$, there is no restriction to communication between players in $N$ and in that case, the Myerson and the Shapley values coincide ($\hat{w}_g = w$).

In a network restricted game the value taken by a coalition depends on the ability of its members to communicate and thus it adopts the point of view of the players. Another perspective is proposed by Meessen [11] and Borm et al. [3]. The focus is not anymore on the role played by the players but rather on the importance of the links to produce some value. In this case, the associated TU game is named the *link game* $(g,\check{w}_g)$ associated with the communication game $(N,w,g)$ and it is defined by

$$\check{w}_g(g') = \sum_{C\in C(g')} w(C) \text{ for all } g' \subseteq g.$$

Let us explain the particularities of this approach. First, the associated TU game $\check{w}$ of a communication game $(N,w,g)$ is defined over subnetworks of the network $g$ : the link game attributes a value to each possible subnetwork of $g$. Second, the value attributed to a given subnetwork $g'$ is the value of the grand coalition given by $w$ and restricted to $g'$. Since the subnetwork $g'$ partitions the grand coalition into components and that no communication between two different components is possible, the value obtainable by the grand coalition is the sum of the values of the components. For a subnetwork $g' \subseteq g$, the set of components of the grand coalition $N$ is $C(g')$. The value in the link game $\check{w}$ of $g'$ is thus the sum of the values of each component $C \in C(g')$.

Borm et al. [3] propose also to adapt the concept of solution to the context of communication games but in using the link game as associated TU game. They call this allocation rule the *position value* for communication games and it is defined as follows :

$$\psi_i^{PV}(N,w,g) = \sum_{l\in L_i(g)} \frac{1}{2}\theta_l^{Sh}(g,\check{w}_g).$$

It is assumed that the characteristic function $w$ of the communication game is zero-normalized.

The interpretation of the position value is quiet easy : we first determine the Shapley value of each link in the associated link game. The Shapley value of each link $ij$ is then shared equally between player $i$ and player $j$. We will come back later to the same kind of approach in another setting.

For an overview of the literature on communication games and other allocation rules the reader is referred to Slikker and van den Nouweland [19].

While communication games use the information given by a network architecture, the values generated in a game are still based on a characteristic function and it is mainly the allocation of value that is affected by the network rather than the generation of productivity values (see Jackson [8]). To allow different network structures to possibly generate different productivity values, Jackson and Wolinsky [9] propose a richer environment : A *network game* is a pair $(N, v)$ where $N$ is the set of players and $v$ is a *value function*[5] that assigns a real value to each possible network on $N$. A value function keeps track of the total value generated by a given network structure (in the same vein a characteristic function specifies a worth for every coalition in TU games). Thus, $v(g)$ represents the worth generated by the players in $N$ while organized through the network $g$. If there is no link between the players, $v(g^{\emptyset}) = 0$ where $g^{\emptyset}$ is the network with no link, it implies that if there is no cooperation between the players then no worth is generated. The set of possible value functions on any network among the player set $N$ is $\mathcal{V}(N)$.

### 2.3 A basis for network games

As for the set of possible TU games, we can define a *basis* for the set of possible network games as a list of network games such that any network game can be expressed as a linear combination of the games in the basis whereas no game in the basis can be expressed as a linear combination of the remaining games in the basis.

Except when otherwise specified in the sequel, the player set $N$ is considered as fixed. In such case, a network game is identified with its value function $v \in \mathcal{V}(N)$.

Let $v_g$ denote the value function defined by

$$v_g(g') = \begin{cases} 1 & \text{if } g \subseteq g' \\ 0 & \text{otherwise.} \end{cases}$$

[5] Slikker [18] calls such a function a *reward function*.

For any $g \subseteq g^N$, we call such value function $v_g$ a *basic value function*. The set of all possible basic value functions forms a basis for network games :

**Proposition 2.1.** *Let $v_g \in \mathcal{V}(N)$ be a basic value function.*

*The set $\left\{v_g \in \mathcal{V}(N) : g \subseteq g^N, g \neq g^\emptyset\right\}$ of all basic value functions forms a basis for $\mathcal{V}(N)$.*

*Proof.* We first note that there exists $(2^{n(n-1)/2} - 1)$ different $v_g$'s. If we view each value function $v \in \mathcal{V}$ as a vector in $\mathbb{R}^{2^{n(n-1)/2}-1}$, we only have to show that any two different $v_g$'s are linearly independent to conclude that the set of $v_g$'s forms a basis of $\mathcal{V}(N)$.

Suppose there exist real numbers $c_g$'s, $g \subseteq g^N$, $g \neq g^\emptyset$,

such that $\sum_{g^\emptyset \neq g \subseteq g^N} c_g v_g(g') = 0$ for all $g' \subseteq g^N$. We prove by induction on $l(g) \equiv \sum_i \frac{1}{2} l_i(g)$ that $c_g = 0$ for all $g \subseteq g^N$.

1. If $l(g) = 1$ then the equivalence $v_{g'}(g) = 1$ if and only if $g = g'$ holds and hence

$$0 = \sum_{g^\emptyset \neq g' \subseteq g^N} c_{g'} v_{g'}(g) = c_g v_g(g) = c_g.$$

   This completes the proof of the induction for $l(g) = 1$.

2. Now, let $2 \leq l(g) \leq n(n-1)/2$ and suppose that $c_{g'} = 0$ for all $g' \subseteq g^N$ with $1 \leq l(g') < l(g)$. For all $g' \subseteq g^N$ with $l(g') \geq l(g)$ we have $v_{g'}(g) = 1$ if and only if $g = g'$. It follows that

$$0 = \sum_{g^\emptyset \neq g' \subseteq g^N} c_{g'} v_{g'}(g) = \sum_{g'; l(g') \geq l(g)} c_{g'} v_{g'}(g) = c_g v_g(g) = c_g.$$

   We conclude that $c_g = 0$ for all $g \subseteq g^N$, $g \neq g^\emptyset$.

   This completes the inductive proof of the linear independence of the set of all basic value functions.

□

What Proposition 2.1 shows is that each value function of a network game can be expressed uniquely as a linear combination of basic value functions : any $v \in \mathcal{V}(N)$ can be decomposed as :

$$v = \sum_{g^\emptyset \neq g \subset g^N} c_g(v) v_g. \tag{2.2}$$

In equation (2.2) we have slightly modified the notation with respect to the notation used in the preceding proof to stress the dependence of $c_g$ to $v$. Since for each value function $v$ corresponds a unique set of scalars $\{c_g\}_{g\subset g^N}$, we stress this dependence by expressing $c_g$ as a function of $v : c_g(v)$.

**Example 1.** *Let $N = \{1,2,3\}$ be the player set and $v \in \mathcal{V}(N)$ defined by*

$$v(g) = \begin{cases} 0 & \text{if } g \in \{g^{\emptyset}, \{12\}, \{23\}\}; \\ 10 & \text{if } g = \{13\}; \\ 16 & \text{if } l(g) = 2 \\ 19 & \text{if } g = g^N. \end{cases}$$

*Then $v$ can be expressed in terms of the basic value functions as follows :*

$$v = 10v_{\{13\}} + 6v_{\{12,13\}} + 16v_{\{12,23\}} + 6v_{\{13,23\}} - 19v_{g^N}.$$

*Indeed, let $g' = \{12, 13\}$ then $v(g') = 10v_{\{13\}}(g') + 6v_{\{12,13\}}(g') = 16$*

Before providing the numerical computation of $\{c_g(v)\}_{g\subset g^N}$ for any value function $v$, the following lemmas are useful.

**Lemma 2.1.** *Let $N$ a (finite) set of players and for any $S \subseteq N$, $g^S$ is the complete network among players in $S$. Then*

$$\sum_{g\subseteq g^S} (-1)^{l(g)} = \begin{cases} 1 & \text{if } S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

*with $l(g)$ the number of links in the network $g$.*

*Proof.* If $S = \emptyset$, then $\sum_{g\subseteq g^S} (-1)^{l(g)} = (-1)^0 = 1$.

Let $S \neq \emptyset$. Pose $l(g^S) = m$. Then,

$$\begin{aligned}\sum_{g\subseteq g^S} (-1)^{l(g)} =&(-1)^0 + \sum_{g:l(g)=1} (-1)^1 + \sum_{g:l(g)=2} (-1)^2 + \cdots + (-1)^m \\ =&\binom{m}{0}(-1)^0 + \binom{m}{1}(-1)^1 + \binom{m}{2}(-1)^2 + \cdots + \binom{m}{m}(-1)^m \\ =&\sum_{k=0}^{m} \binom{m}{k} 1^{m-k}(-1)^k \\ =&(1-1)^m = 0\end{aligned}$$

where the last but one equality follows from the binomial theorem.

□

**Lemma 2.2.** *Let $N$ a (finite) set of players, $S \subseteq N$, $g^S$ the complete network among players in $S$ and $g \subseteq g^S$. Pose $l(g^S) = m$ and $l(g) = p$. Then,*

$$\sum_{g \subseteq g' \subseteq g^S} (-1)^{l(g')} = \begin{cases} (-1)^m & \text{if } g^S = g \\ 0 & \text{otherwise} \end{cases}$$

*with $l(g')$ the number of links in the network $g'$.*

*Proof.* First observe that

$$\sum_{g \subseteq g' \subseteq g^S} (-1)^{l(g')} = \sum_{g'' \subseteq g^S \setminus g} (-1)^{p+l(g'')} = (-1)^p \sum_{g'' \subseteq g^S \setminus g} (-1)^{l(g'')}. \tag{2.3}$$

If $g^S = g$ then equation (2.3) is $(-1)^m(-1)^0$. If $g^S \neq g$ then by lemma 2.1 equation (2.3) is 0. □

We are now in state to present the functional form for any $c_g(v)$, $g \subseteq g^N$, $v \in \mathcal{V}(N)$.

**Proposition 2.2.** *Let $(N, v)$ be a network game with $v \in \mathcal{V}(N)$ and $\{v_g\}_{g \subset g^N}$ the set of basic value functions. Then*

$$v(g') \left( = \sum_{g \subseteq g'} c_g(v) v_g(g) \right) = \sum_{g \subseteq g'} c_g(v) \qquad \text{for all } g' \subseteq g^N \tag{2.4}$$

*if and only if*

$$\left( c_{g'}(v) v_{g'}(g') = \right) c_{g'}(v) = \sum_{g \subseteq g'} (-1)^{l(g')-l(g)} v(g) \qquad \text{for all } g' \subseteq g^N \tag{2.5}$$

*Proof.* Simple algebraic computations from lemmas 2.1 and 2.2 will prove both implications.

1. Suppose that equation (2.4) holds. Then

$$\begin{aligned}
\sum_{g \subseteq g'} (-1)^{l(g')-l(g)} v(g) &= (-1)^{l(g')} \sum_{g \subseteq g'} (-1)^{l(g)} v(g) \\
&= (-1)^{l(g')} \sum_{g \subseteq g'} (-1)^{l(g)} \sum_{g'' \subseteq g} c_{g''}(v) \\
&= (-1)^{l(g')} \sum_{g'' \subseteq g'} c_{g''}(v) \sum_{g'' \subseteq g \subseteq g'} (-1)^{l(g)} \\
&= (-1)^{l(g')} c_{g'}(v) (-1)^{l(g')} = c_{g'}(v).
\end{aligned}$$

2. Suppose now that equation (2.5) holds. Then

$$\begin{aligned}\sum_{g\subseteq g'} c_g(v) &= \sum_{g\subseteq g'}\sum_{g''\subseteq g}(-1)^{l(g'')-l(g)}v(g'')\\ &= \sum_{g''\subseteq g'}(-1)^{l(g'')}v(g'')\sum_{g''\subseteq g\subseteq g'}(-1)^{l(g)}\\ &=(-1)^{l(g')}v(g')(-1)^{l(g')} = v(g'). \end{aligned} \tag{2.6}$$

□

**Example 2.** *Let $(N, v)$ be the network game as defined in example 1. Then*

$$c_{\{12,13\}}(v) = (-1)^{2-1}v(\{12\}) + (-1)^{2-1}v(\{13\}) + (-1)^{2-2}v(\{12, 13\}) = 0 - 10 + 16 = 6.$$

## 2.4 Allocation rules for network games

Thanks to a value function $v \in \mathcal{V}(N)$, we know the total value generated by each possible network $g$ on a player set $N$. How should we redistribute the worth of the network collaboration among players ? This is described by an allocation rule :

**Definition 2.1.** *An allocation rule is a function $Y : G(N)\times\mathcal{V}(N) \to \mathbb{R}^n$ such that $\sum_i Y_i(g, v) = v(g)$ for all $v \in \mathcal{V}(N)$ and all $g \subseteq g^N$.*

Each $Y_i(g, v) \in \mathbb{R}$ is interpreted as the payoff player $i$ receives by collaborating in the network $g$ whose worth is given by $v(g)$. Note that the definition of the allocation rule forces the value of a given network $g$ to be fully redistributed among the players. Jackson [7] calls this condition *balancedness*.

As shown by Jackson and Wolinsky [9], the Shapley value and the Myerson value for TU games and communication games respectively, have a natural extension as allocation rule for network games. They call this allocation rule the Myerson value for network games and can be expressed as follows :

$$Y_i^{MV}(g, v) = \sum_{S\subset N\setminus\{i\}} (v(g|_{S\cup\{i\}}) - v(g|_S))\left(\frac{|S|!(n - |S| - 1)!}{n!}\right). \tag{2.7}$$

The Myerson value for network games (2.7) presents some similarities with the Shapley value (2.1). Both are weighted averages of marginal contributions, but in (2.1) the marginal contributions are over the value of coalitions (given by a characteristic function $w$), whereas in (2.7) marginal contributions are over the value of networks (tracked by a value function $v$). Note also that the Myerson value as allocation rule depends on both a value function and a given network $g$.

As we did for the Myerson value for communication games, we can define the Myerson value for network games as a solution for a corresponding TU game. Let $v$ be a value function in $\mathcal{V}(N)$ and $g \in G(N)$ a network. In such a case, players in a coalition $S \subset N$ can use the links in $g|_S$ to communicate and generate the value $v(g|_S)$. From any triple $(N, v, g)$ we can associate a particular TU game $(N, w_g^v)$ such that $w_g^v(S) = v(g|_S)$ for all $S \subset N$. Hence, the Myerson value as allocation rule defined in (2.7) can be expressed as

$$Y_i^{MV}(v, g) = \theta_i^{Sh}(w_g^v) \tag{2.8}$$

where $\theta_i^{Sh}(.)$ is the Shapley value as defined in (2.1).

**Example 3.** *Let $N = \{1, 2, 3\}$ the player set, $g' = \{13, 23\}$ a network and the value function $v \in \mathcal{V}(N)$ defined by*

$$v(g) = \begin{cases} 0 & \text{if } g \in \{g^{\emptyset}, \{12\}, \{23\}\}; \\ 10 & \text{if } g = \{13\}; \\ 16 & \text{if } l(g) = 2 \\ 19 & \text{if } g = g^N. \end{cases}$$

*The TU game $(N, w_g^v)$ associated with the value function $v$ and the network $g' = \{13, 23\}$ is*

$$w_{g'}^v(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\emptyset, \{12\}, \{23\}\}; \\ 10 & \text{if } S = \{1, 3\}; \\ 16 & \text{if } S = N. \end{cases}$$

*where for example $w_{g'}^v(N) = 16$ by $v(g'|_N) = v(\{13, 23\}) = 16$.*

*By equations (2.8) and (2.1) we find that $Y^{MV}(v, g') = \theta^{Sh}(w_{g'}^v) = (7, 2, 7)$. Similarly we find that $Y^{MV}(v, g^N) = \theta^{Sh}(w_{g^N}^v) = (8, 3, 8)$ by $v(\{13, 23\}) = 19$.*

*See the Appendix for details.*

Since any value function $v$ can uniquely be decomposed in terms of basic value functions $v_g$'s with appropriate unanimity coefficients $c_g(v)$'s (see Proposition 2.2), there exists an alternative way to compute the Myerson value as allocation rule for network games. The following proposition (due to Slikker and van den Nouweland [19] p.119) shows this alternative computation :

**Proposition 2.3.** *Let $(N, v)$ be a network game. Then for any network $g \subseteq g^N$ the Myerson value for player i is :*

$$Y_i^{MV}(g, v) = \sum_{g' \subseteq g : i \in N(g')} \frac{c_{g'}(v)}{n(g')} \tag{2.9}$$

*with $n(g')$ the number of players involved in the network $g'$.*

The Myerson value shares equally between the players members of the sub-networks of the network under consideration the values given by the unanimity coefficients.

**Example 4.** *Let $(N, v)$ be the network game defined in examples 1 and 3. We know that the decomposition of the value function v is*

$$v = 10v_{\{13\}} + 6v_{\{12,13\}} + 16v_{\{12,23\}} + 6v_{\{13,23\}} - 19v_{g^N}.$$

*We now compute the Myerson value using the expression (2.9) with the network $g' = \{13, 23\}$ :*

$$Y_1^{MV}(g', v) = \frac{10}{2} + \frac{0}{3} + \frac{0}{3} + \frac{6}{3} - \frac{0}{3} = 7$$
$$Y_2^{MV}(g', v) = \frac{0}{2} + \frac{0}{3} + \frac{0}{3} + \frac{6}{3} - \frac{0}{3} = 2$$
$$Y_3^{MV}(g', v) = \frac{10}{2} + \frac{0}{3} + \frac{0}{3} + \frac{6}{3} - \frac{0}{3} = 7.$$

*Again we find the same result as computed in example 3.*

An allocation rule is a way to allocate the total value generated by a set of players collaborating in a two-by-two basis. From a normative point of view, the allocation rule has to take into account the marginal value of a player, that is, the value added to the network by the player's participation. Moreover the allocation rule should also take into account the bargaining power of a player : what would happen to the total productivity of a network if the player were to break one or several of her links ? Players control the links and decide which link to create or to sever. Moreover, the value generated by any network is due to the presence or the absence of the possible links between players. For these reasons, the redistribution of the total value could be done in terms of the links controlled by the players. This leads to the following definition :

**Definition 2.2.** *An allocation rule Y is linked-based if there exists a function* $\psi : G(N)\times\mathcal{V}(N) \to \mathbb{R}^{n(n-1)/2}$ *such that* $\sum_{ij\in g}\psi_{ij}(g,v) = v(g)$*, and*

$$Y_i(g,v) = \sum_{j\neq i}\frac{\psi_{ij}(g,v)}{2}.$$

The total productivity of a network is indirectly allocated to the players : the value is first distributed to the links and then to the players controlling the links. Note that in this definition, we consider that both players in a given link are equally important to form and maintain the link. We assume that the value of the link is equally distributed among the two players.[6]

The position value defined for communication games (see section 2.2) follows the same logic. Similarly to the Myerson value, we can adapt the position value as a link-based allocation rule for network games. The position value for network games can be expressed as follows :

$$Y_i^{PV}(g,v) = \sum_{j\neq i:\, j\in N(g)}\left[\sum_{g'\subseteq g\setminus ij}\frac{1}{2}\left(v(g'+ij)-v(g')\right)\left(\frac{l(g')!(l(g)-l(g')-1)!}{l(g)!}\right)\right]. \quad (2.10)$$

Note that the first summation is over the players members of the network $g$ and the second summation is over all subnetworks of the network $g$ excluding the link $ij$. As usual, the allocation rule strongly depends on the network $g$ that is formed. The interpretation of the position value as allocation rule for network games is similar to the Myerson value for network games given its Shapley-style calculations and allocates value based on those calculations. We can think of building up the network $g$ by adding links one by one and then seeing what value is generated through this process. Players are allocated a payoff given by the marginal contributions to the overall value of the links they are involved in.

A less tedious calculation makes use of the decomposition of the value function $v$ in terms of basic value functions, as in expression 2.9 for the Myerson value. The following proposition is due to Slikker and van den Nouweland [19].

**Proposition 2.4.** *Let* $(N,v)$ *be a network game. Then for any network* $g\subset g^N$ *the*

[6] This is the same assumption as in Borm et al. Nevertheless, it could be interesting to explore some general rule of distribution of the value of the link, for example when the bargaining power of the players is different (see e.g. Kamijo and Kongo [10]).

*Position value for player i is :*

$$Y_i^{PV}(g,v) = \sum_{g' \subset g} \frac{c_{g'}(v) l_i(g')}{2l(g')} = \sum_{g' \subset g} \sum_{l \in L_i(g')} \frac{c_{g'}(v)}{2l(g')}. \tag{2.11}$$

**Example 5.** *Let $(N,v)$ be the network game defined in examples 1, 3 and 4 with*

$$v = 10v_{\{13\}} + 6v_{\{12,13\}} + 16v_{\{12,23\}} + 6v_{\{13,23\}} - 19v_{g^N}.$$

*We now compute the position value using the expression (2.11) with the network $g' = \{13,23\}$ :*

$$\begin{aligned}
Y_1^{PV}(g',v) &= \frac{10\times 1}{2\times 1} + 0 + 0 + \frac{6\times 1}{2\times 2} - 0 &&= \frac{13}{2}\\
Y_2^{PV}(g',v) &= \frac{10\times 0}{2\times 1} + 0 + 0 + \frac{6\times 1}{2\times 2} - 0 &&= \frac{3}{2}\\
Y_3^{PV}(g',v) &= \frac{10\times 1}{2\times 1} + 0 + 0 + \frac{6\times 2}{2\times 2} - 0 &&= 8.
\end{aligned}$$

*Observe that for the same network game, the position value and the Myerson value do not coincide.*

For more discussion and definitions about network games and allocation rules, the reader is refereed to Jackson [6] or Jackson [8] and Slikker and van den Nouweland [19] chap. 4.5.

## 2.5 Link games associated with value functions

So far we have defined two different types of games, one which assigns value to any *coalition* or subset of players and one which assigns value to any network of players. It is well known that the second type of games is a richer object than the first type of games. This is due to the fact that in TU games, collaboration is transitive : if a player $i$ collaborates with a player $j$, and if player $j$ collaborates with a player $k$, all three players belong to the same coalition. Thus, whether player $i$ directly collaborates with player $k$ or not has no impact on the value of their collaboration. In the context of networks, intransitivities are allowed. In a setting where players $i$ and $j$ collaborate, and so do players $j$ and $k$ but not $i$ and $k$ will generally generate a different productivity value than the structure in which all three players jointly collaborate. This can be directly seen in their respective domain of definition. For a given set of players $N$, TU games are defined on a $2^n$-dimension domain and network games are defined on a $2^{n(n-1)/2}$-dimension domain. Nevertheless, as Slikker [18] puts it, network games can be seen as TU-games : the value function $v$ of the network game $(N, v)$ can be seen as the characteristic function of the TU game $(g^N, v)$. The trick is to consider the set of links as the new set of players and using the value function as the characteristic function of the new game. Slikker calls this TU game the *link game* associated with the network game $(N, v)$. This operation mimics the building of an edge-graph[7] from a given graph where each edge of the original graph is a vertex in the new graph : players in the associated link game are the links of the network game under consideration.

**Definition 2.3.** *Let $(N, v)$ be a network game with value function $v$. The associated link game of $(N, v)$ is the TU game $(g^N, v)$ where the set of players is the set of possible links $g^N$ and the characteristic function is the value function $v$ of the network game.*

**Example 6.** *Let $N = \{1, 2, 3\}$ and $v \in \mathcal{V}(N)$ be such that $v(g) = l(g)$ for any $g \subseteq g^N$. Then, the associated link game $(g^N, v)$ is such that $v(S) = |S|$ for any $S \subseteq g^N$, where $S$ denotes a set of links.*

## 3 Imputations and the core of network games

In this section we adapt some important solution concepts from TU games to the context of network games. Once we know the total productive value of the different possible networks, a natural question that arises is how to distribute the value of these productive collaborations among the players. We have seen in section 2 how to answer this question thanks to an allocation rule, that defines an allocation $\mathbf{Y} \in \mathbb{R}^{\mathbf{N}}$ for each possible value function $v$ and network $g$. We now turn on to different motivations that can be put forward. We define a vector whose components indicate how the total value is distributed, but we relax the assumptions made in the definition of an allocation rule and impose some other conditions. We first begin with some stability consideration.

**Definition 3.1.** *Let $(N, v)$ be a network game. A vector $\mathbf{x} \in \mathbb{R}^{n(n-1)/2}$ is called a* **link-based imputation for network game** *if*

[7] Also called adjoint graph, derived graph, ... see Balakrishnan [1].

1. $\mathbf{x}$ *is link rational :*

$$x_{ij} \geq v(ij) \qquad \textit{for all } ij \in g^N,$$

2. $\mathbf{x}$ *is efficient :*

$$\sum_{ij \in g^N} x_{ij} = v(g^N).$$

We denote the set of link-based imputations of $(N, v)$ by $I(v)$ and any element $\mathbf{x}$ of $I(v)$ is a payoff distribution of the total worth of the complete network $g^N$ which gives each link $l \equiv ij$, $i \neq j$, a payoff $x_{ij}$ greater or equal than the value the link can guarantee to itself if it is the only link in the network. Note that the set of imputations for a network game with value function $v$ is nonempty if and only if

$$v(g^N) \geq \sum_{l \in g^N} v(l).$$

An imputation indicates how the value of the complete network is allocated among the links. As it is the case for imputations in TU games, it is thus presumed that the complete network among the player set $N$ generates the maximum possible value. The equivalent concept to link rationality in the context of coalitional TU games is *individual rationality*, according to which each player should receive at least the payoff he would receive when he operates alone, i.e. as singleton. In the context of network games, a value can be generated only if at least two players decide to bilaterally cooperate. The notion of marginal value generated is thus better captured in terms of links rather than in terms of players,hence in the context of network games, we found it more natural to use the notion of link rationality rather than individual rationality.

Which is the relationship between the set of imputations and the set of allocation rules? Let $\mathbf{x}$ be an imputation for a network game $(N, v)$. Construct $Y_i = \sum_{j \neq i} \frac{x_{ij}}{2}$. We have no guarantee that $\sum_{i \in N} Y_i = v(g)$ for all $g \subseteq g^N$ and hence $Y$ may fail to be an allocation. Now let $Y_i(g, v)$ be a link-based allocation as described in definition 2.2. Let $\mathbf{x} = \{x_{ij}\}_{ij \in g^N}$ be such that

$$x_{ij} = \frac{Y_i(g, v)}{l_i(g)} + \frac{Y_j(g, v)}{l_j(g)}.$$

But then, we have no guarantee that $x_{ij} \geq v(ij)$ for all $ij \in g^N$, and $\mathbf{x}$ is not an imputation. The reason why the set of imputations is different than the set of allocations is the local properties of imputations : an imputation only requires link rationality and the distribution of the value of the complete network. However, an allocation rule specifies how to distribute the value of any network. Hence, we see that if $g = g^N$, then an imputation is a (link-based) allocation.

We now restrain the set of admissible imputations to payoff distributions that are *coalitionally* rational. Again, we have a network game $(N, v)$ and a vector $\mathbf{x} \in \mathbb{R}^{n(n-1)/2}$. We say that a vector $\mathbf{x}$ is a *feasible* payoff distribution for a coalition $S$ of players ($S \subseteq N$) if and only if for any network $g \subseteq g^S$, we have :

$$\sum_{l \in g} x_l \leq v(g).$$

Hence, the players in coalition $S$ can collaborate together in a network $g$ and distribute the worth $v(g)$ among their links as prescribed by the components of $\mathbf{x}$. A payoff distribution is simply *feasible* if it is feasible for the grand coalition $N$. We say that a coalition of players $S \subseteq N$ can improve on a payoff vector $\mathbf{x}$ if and only if $v(g) > \sum_{l \in g} x_l$ for any $g \subseteq g^S$. This last condition says that there exists another vector $\mathbf{y} \in \mathbb{R}^{n(n-1)/2}$, $\mathbf{y} \neq \mathbf{x}$ where $\mathbf{y}$ is feasible for $S$ and such that the links of the players in $S$ get a strictly higher payoff. We say that the payoff distribution $\mathbf{y}$ *dominates* $\mathbf{x}$ or that the payoff distribution $\mathbf{x}$ is *dominated* by $\mathbf{y}$. The *core* of a network game $(N, v)$ is simply the set of feasible payoff distribution vectors that are undominated, that is, there is no coalition for which there exists an alternative feasible payoff distribution that gives to every link a better payoff.

**Definition 3.2.** *Let $(N, v)$ be a network game. The core of the game $(N, v)$ is the set*

$$C(v) \equiv \left\{ \mathbf{x} \in I(v) : \sum_{l \in g} x_l \geq v(g) \text{ for all } g \subseteq g^N,\ g \neq g^{\emptyset} \right\}.$$

The core is a very important stability concept for TU games. For network games, if a payoff distribution $\mathbf{x}$ is not in the core of the game, then it is much a matter of fairness or efficiency than stability. If an allocation $\mathbf{x}$ is not in the core of the network game, it means that there exists a network $g \neq g^N$ such that $\sum_{l \in g} x_l < v(g)$. Since $v(g)$ can be interpreted as the stand-alone value of $g$, the value generated only by the links in $g$, an allocation $\mathbf{x}$ outside the core redistributes less to the links in $g$ than what these links would have produced on their own. Due to the possible presence of externalities across links, the core of a network game can be empty.

**Example 7.** *Let* $N = \{1,2,3\}$ *and* $v \in \mathcal{V}(N)$ *:* $v(\{12\}) = v(\{13\}) = v(\{23\}) = 0$, $v(\{12,23\}) = 4$, $v(\{12,13\}) = 3$, $v(\{13,23\} = 2$ *and* $v(g^N) = 4$. *Then the core of* $(N,v)$ *is empty, since solving the set of linear inequalities given by definition 3.2, we have* $2(x_{12} + x_{23} + x_{13}) \geq 9$ *which is incompatible with the efficiency requirement* $\sum_{l \in g^N} x_l = 4$.

By definition 2.2 imposing efficiency of an allocation rule and definition 3.2, if the core of a network game is nonempty then a link-based allocation is in the core of the game. The characterization of the class of network games with nonempty cores is based on a characterization due to Bondareva [2] and Shapley [16] for the class of coalitional TU games in which the core is nonempty. Their characterization relies on duality theory of linear programming. To understand this core existence problem, we can restate the definition of the core of a network game as follows : what is the minimum total payoff that guarantees that each possible network receives at least its stand-alone value $v(g)$ ? In mathematics this question corresponds to the following linear programming problem :

The objective is to

$$\min_{\mathbf{x} \in \mathbb{R}^{n(n-1)/2}} \sum_{l \in g^N} x_l \tag{3.1}$$

with the linear constraints

$$\sum_{l \in g} x_l \geq v(g) \text{ for all } g \subseteq g^N,\ g \neq g^{\emptyset}. \tag{3.2}$$

We remind that we denote by $G(N)$ (or simply $G$) the set of all possible networks among the players in $N$, that is

$$G \equiv \{g : g \subseteq g^N\}$$

and the cardinality of $G$ is denoted $|G|$.

The dual of the linear program (3.1)-(3.2) is thus :

$$\max_{\lambda(g) \in \mathbb{R}_+^{|G|}} \sum_{g \subseteq g^N} \lambda(g) v(g) \tag{3.3}$$

subject to

$$\sum_{g \supset l} \lambda(g) = 1, \text{ for all } l \in g^N. \tag{3.4}$$

Before we provide the theorem stating core non-emptiness, we provide some definitions of concepts derived from the dual program.

Let $N$ be the finite set of players. A map $\lambda : G \to \mathbb{R}_+$ is called a *balanced map* if

$$\sum_{g \subseteq g^N} \lambda(g)\mathbf{e}^g = \mathbf{e}^{g^N}$$

where $e^g$ is a characteristic vector for network $g$ with components

$$\mathbf{e}^g_l = 1 \text{ if } l \in g \text{ and } \mathbf{e}^g_l = 0 \text{ if } l \in g^N \setminus g.$$

A collection $B$ of networks is called *balanced* if there is a balanced map $\lambda$ such that $B = \{g \in G : \lambda(g) > 0\}$.

A network game $(N, v)$ with $v \in \mathcal{V}(N)$ is *balanced* if for each balanced map $\lambda : G \to \mathbb{R}_+$, we have

$$\sum_{g \subseteq g^N} \lambda(g)v(g) \leq v(g^N).$$

**Theorem 3.1.** *Let $(N, v)$ be a network game with $v \in \mathcal{V}(N)$. Then the following two assertions are equivalent :*

*(i)* $C(v) \neq \emptyset$.

*(ii)* $(N, v)$ *is a balanced network game.*

*Proof.* The non-emptiness of the core in $(i)$ is satisfied if and only if the linear program (3.1)-(3.2) is satisfied and the optimal value is $v(g^N)$. By the duality theorem (see *e.g.* Franklin [5], p.80), $v(g^N)$ is the optimal value of the dual program (3.3)-(3.4), as both programs are feasible. Hence, $(N, v)$ has a nonempty core if and only if

$$v(g^N) \geq \sum_{g \subseteq g^N} \lambda(g)v(g) \tag{3.5}$$

which states the equivalence between $(i)$ and $(ii)$. □

# 4 Simple network games

In this section we introduce a special class of network games : the *simple network games*. A network game is *simple* if any possible network takes the value of 1 or

0.[8] Formally :

**Definition 4.1.** *Let $(N, v)$ be a network game. $(N, v)$ is a simple network game if and only if*

$$v(g) \in \{0, 1\}$$

*for all $g \subseteq g^N$ and*

$$v(g^N) = 1.$$

A simple network game describes a situation where some networks are *winning* while the other networks are *nonwinning*. That is, only certain configurations can achieve positive value (of 1) according to the value function of the network game. In such a case, a simple network game $(N, v)$ can equally be described by its set of *winning networks* : $W(v) \equiv \{g \subseteq g^N : v(g) = 1\}$. The networks in $G$ that are not a member of $W(v)$ take the value 0. The complete network always has the value 1.

Some players in a network game can play a special role in the determination of the value taken by networks. For example, the presence of certain players may be essential in a network to achieve some positive value. This kind of player is called a *veto player*.

**Definition 4.2.** *A player $i$ is a veto player for a network game $(N, v)$ if and only if*

$$v(g) = 0 \qquad \text{for any } g \subseteq g^{N \setminus \{i\}}.$$

This means that the cooperation of the veto player $i$ is required to obtain profits, and any network without $i$ achieves 0 value. We say that a network game is a *network game with veto control* if there is at least one veto player.

Accordingly, we may imagine some network games that can achieve some non-negative value if and only if we observe the collaboration between two specific players, that is the presence of a given link is necessary. We call this kind of link an *essential link*.

**Definition 4.3.** *A link $l$ is an essential link for a network game $(N, v)$ if and only if*

$$v(g) = 0 \qquad \text{for any } g \subseteq g^N \setminus \{l\}.$$

[8] Sometimes in simple TU games, the condition of monotonicity is also added to the definition of simple game : supersets of any coalition with value 1 have a value of 1.

If we want a network to be fair in the sense that each network receives at least its stand-alone value, we have to chose a core allocation. In the following theorem we show that the core of a simple network game is nonempty if and only if the game has at least **one essential link**.

**Theorem 4.1.** *Let* $(N, v)$ *be a simple network game. Then, the core of* $(N, v)$ *is nonempty if and only if there is at least one essential link.*

*Proof.* For each link $l \in g^N$, let $\mathbf{e}^l \in \mathbb{R}^{n(n-1)/2}$ denote the vector with the $l$-th coordinate equal to 1 and all other coordinates are 0. To show that the core of a simple network game is nonempty we have to show that there exists at least one essential link. The proof consists in proving the following equivalence : the core of a simple game $(N, v)$ is

$$C(v) = \mathcal{H}\left\{\mathbf{e}^l \in \mathbb{R}^{n(n-1)/2} : l \text{ is an essential link for } v\right\}$$

with $\mathcal{H}$ meaning the convex hull. The core of the simple game $C(v)$ is thus a compact convex polyhedron consisting of all points in $\mathbb{R}^{n(n-1)/2}$ that can be expressed as a convex combination of the extreme points $\mathbf{e}^l$.

1. Suppose that $l$ is an essential link for $v$. Let $g \subseteq g^N$. If $l \in g$ then $\sum_{l' \in g} e^l_{l'} = 1 \geq v(g)$, otherwise $\sum_{l' \in g} e^l_{l'} = 0 = v(g)$. We know that $v(g^N) = 1 = \sum_{l' \in g^N} e^l_{l'}$. So $\mathbf{e}^l$ is in $C(v)$. This prove the inclusion $\supseteq$ because $C(v)$ is a convex set.[9]

2. Now, let $\mathbf{x} \in C(v)$. It is sufficient to prove that any non-essential link must be allocated 0 according to $\mathbf{x}$. Suppose on the contrary that $l \in g$ is a non-essential link and that $x_l > 0$. Consider $g$ with $v(g) = 1$ and $l \notin g$. Note that such winning network $g$ must exist, because otherwise, $l$ would be essential. Then
$$\sum_{l' \in g} x_{l'} = \sum_{l' \in g^N} x_{l'} - \sum_{l' \in \{g^N \setminus g\}} x_{l'} \leq 1 - x_l < 1$$
contradicting that $\mathbf{x}$ is a core element.

□

**Corollary 4.1.** *Any simple network game* $(N, v)$ *with veto control has non-empty core.*

[9] The core of network games is defined on a hyperplan by loose inequalities, and since the value is freely transferable, the core is a convex set.

*Proof.* the proof is very simple by remarking that any link involving the veto player is an essential link. □

The last theorem shows that the core of any simple network game is nonempty if and only if there is an essential link, but not necessary a veto player. It is thus a weaker condition then in caolitional simple TU games where the presence of a veto player is necessary for the non-emptiness of the core of coalitional TU games.

# 5 The link-based Shapley value for network games

The core of network games shows how to redistribute the productive value of the complete network such that no subset of links would be better off, in terms of achieving a higher value in other network. But still the core of a network can be empty or if not, does not single out an allocation. In most cases where the core is nonempty, the core is set-valued. Another property widely used in the context of network games is to ask the allocation to be *component balanced*. Let $v$ be a value function. We say that $v$ is *component additive* if $\sum_{h \in C(g)} C(h) = v(g)$, that is if the value function rules out externalities across components : the value taken by a component is independent on how the other components are structured. For any component additive value function, a component balanced allocation rule allocates the value of each component among the connected players. The Myerson value (expression 2.7) and the position value (expression 2.10) as allocation rules for network games are examples of component balanced allocation rules. In a sense, component balancedness can be seen as a restriction of the core property. Within the core, we care about any coalitional deviations, whereas with component balancedness, only deviations by coalitions of connected players are contemplated. But, as Jackson [7] puts it (p.138), component balancedness is not sufficient to ensure that an allocation lies in the core of a network game, but is strong enough to be in conflict with some fairness and anonymity properties.

The aim of this section is to propose a payoff allocation based on some desirable properties. The Shapley value is a one-point solution concept for TU games which is the only one to satisfy a number of properties or axioms. The Shapley value is well documented in the context of TU games, it has been rediscovered and/or re-characterized many times and has been used in many applications. For these reasons, we found interesting to see whether such solution concept could be transposed to network games using the associated link-game interpretation we have developed for any network game.

**Definition 5.1.** *For any network game* $(N, v)$ *with* $v \in \mathcal{V}(N)$*, a solution* $\phi : \mathcal{V}(N) \to \mathbb{R}^N$ *is* **link-based** *if there exists an associated function* $\psi : \mathcal{V}(N) \to \mathbb{R}^{n(n-1)/2}$ *such that* $\sum_{ij \in g^N} \psi_{ij} = v(g^N)$ *and*

$$\phi_i(v) = \sum_{j \neq i} \frac{\psi_{ij}(v)}{2}.$$

Note that in the definition 5.1 a **solution** is defined over the set of possible value functions, whereas an allocation rule depends on both the value function and the network. The following properties will help us to characterize the Shapley value as link-based solution in the context of network games. They are the natural transpositions of the original axioms proposed by Shapley [15] for TU games.[10] In the sequel, we consider the player set $N$ as fixed and the network game (or game) is designated by its value function.

**Axiom 5.1** (Efficiency)**.** *For any network game* $(N, v)$*, with* $v \in \mathcal{V}(N)$*,* $\phi$ *is an efficient link-based solution if and only if for its associated function* $\psi$ *:*

$$\sum_{ij \in g^N} \psi_{ij}(v) = v(g^N)$$

*or equivalently*

$$\sum_{i \in N} \phi_i = v(g^N).$$

This property means that a solution for a network game is a feasible vector of real numbers such that the sum of its components is $v(g^N)$. This is the direct transposition of the *efficiency* axiom in the context of TU games where it is generally assumed that the grand coalition generates the maximum possible value. Note that this property is different than the balancedness condition imposed in the definition of an allocation rule. Here, whatever the network the players end up with, the value distributed is the one of the complete network. On the contrary, a component balanced allocation rule only distributes the value of the given network that actually forms. For this reason, the characterization of the Shapley value as a (link-based)

[10]To be precise, we follow the more popular characterization without using the notion of *carrier*.

solution for network games we offer here is different than the characterization of the Shapley value as allocation rule (see Jackson and Wolinsky [9]).

We say that a link $ij$ is *superfluous* in a game $(N,v)$ if $v(g+ij) = v(g)$ for every network $g \subseteq g^N$. This means that the presence of a superfluous link does not affect or contribute anything to any network. Hence, it seems natural that any solution should attribute a zero payoff to such links :

**Axiom 5.2** (Superfluous link property)**.** *For any network game $(N,v)$, with $v \in \mathcal{V}(N)$, $\psi_{ij}(v) = 0$ for all superfluous links $ij \in g^N$.*

We say that two links $l$ and $l'$ are symmetric in the game $(N,v)$ if $v(g+l) = v(g+l')$ for any network $g \subseteq g^N \setminus \{l,l'\}$. Symmetric links contribute the same amount to any network. Therefore, it seems natural that a solution gives symmetric links the same payoff :

**Axiom 5.3** (Symmetry)**.** *For any network game $(N,v)$, with $v \in \mathcal{V}(N)$, $\psi_l(v) = \psi_{l'}(v)$ for all symmetric links $l,l' \in g^N$ in $v$.*

The last property needed in the characterization may be seen as a consistency or decomposability condition. The way the value of a network is allocated may be decomposed so that one may separately allocate the value on different parts of the value function and then sum up :

**Axiom 5.4** (Additivity)**.** *For any two network games $(N,v)$ and $(N,v')$, with $v \in \mathcal{V}(N)$ and $v' \in \mathcal{V}(N)$ , $\psi_l(v+v') = \psi_l(v)+\psi_l(v')$ for any link $l \in g^N$ with $(v+v')(g) \equiv v(g)+v'(g)$.*

**Definition 5.2.** *The Shapley value of a network game $(N,v)$ is a* **link-based** *solution $\phi^{Sh} : \mathcal{V}(N) \to \mathbb{R}^N$ with an associated function $\psi^{Sh}$ that satisfies*

$$\psi_{ij}^{Sh}(v) = \sum_{g \subset g^N \setminus \{ij\}} \frac{l(g)!\,([n(n-1)/2]-l(g)-1)!}{[n(n-1)/2]!}\,(v(g+ij)-v(g)) \tag{5.1}$$

*and*

$$\phi_i^{Sh}(v) = \sum_{j \neq i} \frac{\psi_{ij}(v)}{2}.$$

As for the Myerson and position values as allocation rules, the link-based Shapley value can also be expressed using the unanimity coefficients :

**Definition 5.3.** *The Shapley value of a network game* $(N, v)$ *is defined by :*

$$\phi_i^{Sh}(v) = \sum_{g \subset g^N} \frac{c_g(v) l_i(g)}{2l(g)} = \sum_{g \subset g^N} \sum_{l \in L_i(g)} \frac{c_g(v)}{2l(g)}. \tag{5.2}$$

**Theorem 5.1.** *Let* $\phi : \mathcal{V}(N) \to \mathbb{R}^N$ *be a* **link-based** *solution for network games with an associated function* $\psi : \mathcal{V}(N) \to \mathbb{R}^{n(n-1)/2}$*. Then* $\psi$ *satisfies efficiency, superfluous link property, symmetry and additivity if and only if* $\phi$ *is the link-based Shapley value* $\phi^{Sh}$*.*

*Proof.* We can see that the Shapley value satisfies the four properties. Additivity follows from the linear form of expression (5.1).

The superfluous link property is also satisfied by (5.1) since $v(g+ij) - v(g)$ is always zero for a superfluous link. If two links $l$ and $l'$ are symmetric then $v(g+l) - v(g) = v(g+l') - v(g)$ for all $g$. Finally, efficiency is satisfied by observing that $\sum_{l \in g^N} \sum_{g \subset g^N \setminus \{l\}} \frac{l(g)!([n(n-1)/2] - l(g) - 1)!}{[n(n-1)/2]!} = 1$.

We have to show that $\phi^{Sh}$ is the only link-based solution to network games such that the associated function $\psi^{Sh}$ satisfies the four properties.

Let $\phi$ be a link-based solution such that the associated function $\psi$ satisfies the four properties. We want to show that $\phi = \phi^{Sh}$. Let $v \in \mathcal{V}(N)$. By Proposition 2.1, there exist a unique set of numbers $c_g$ $(g \neq g^{\emptyset})$ such that $v = \sum_{g \neq g^{\emptyset}} c_g v_g$. By additivity of the associated functions, it follows that

$$\psi(v) = \sum_{g \neq g^{\emptyset}} \psi(c_g v_g)$$

and

$$\psi^{Sh}(v) = \sum_{g \neq g^{\emptyset}} \psi^{Sh}(c_g v_g).$$

Hence, it is sufficient to show that for any $g \subseteq g^N$, $g \neq g^{\emptyset}$ and a scalar $c$ :

$$\psi(c v_g) = \phi^{Sh}(c v_g). \tag{5.3}$$

Chose $g \neq g^{\emptyset}$ and a scalar $c$. Then, for any link $ij \notin g$ :

$$c v_g(g' + ij) - c v_g(g') = 0 \qquad \text{for all } g',$$

so that $ij$ is superfluous in $cv_g$. By the superfluous link property of $\phi$ and $\phi^{Sh}$ :

$$\psi_{ij}(c v_g) = \psi_{ij}^{Sh}(c v_g) = 0 \qquad \text{for all } ij \notin g. \tag{5.4}$$

Now suppose that $l$ and $l'$ are links in the network $g$. Then, for any network $g' \subseteq g^N \setminus \{l, l'\}$,

$$cv_g(g' + l) = cv_g(g' + l') = 0$$

implying that links $l$ and $l'$ are symmetric in the network game $cv_g$. Hence by the symmetry property of the associated function of link-based solutions $\phi$ and $\phi^{Sh}$ :

$$\phi_l^{Sh}(cv_g) = \phi_k^{Sh}(cv_g) \qquad \text{for all links } l \text{ and } l' \text{ in } g \tag{5.5}$$

and

$$\psi_l(cv_g) = \psi_{l'}(cv_g) \qquad \text{for all links } l \text{ and } l' \text{ in } g. \tag{5.6}$$

By efficiency, (5.4), (5.5) and (5.6) we obtain

$$\phi_l^{Sh}(cv_g) = \phi_l(cv_g) = \frac{c}{l(g)} \qquad \text{for all links } l \in g. \tag{5.7}$$

and now, (5.4) and (5.7) imply (5.3). □

We thus see that defined in the context of network games with value function, the adaptation of the axioms proposed in Borm et al. [3] are necessary and sufficient to characterize the link-based Shapley value, called the position value in the context of communication games.

# 6 Comparisons between the Myerson and position values and the link-based Shapley value

Given the similarities of the Myerson value and the position value adapted to network games, the position value inherits the same kind of problems stressed by Jackson [7] for the Myerson value. From a network formation perspective, the position value does not properly account for other possible networks. If alternative network structures are available, then values of all possible alternative networks should be taken into account, and not just subnetworks, in determining the allocation. If the network is seen as a given fixture, then it is not clear why the value of subnetworks should enter allocation computations. With the link-based Shapley value we propose, we answer partly the first problem. Indeed, by the efficiency axiom, the value of the complete network is allocated, and the value of all possible subnetworks are taken into account. Thus, the link-based Shapley value only coincides with the position value for the complete network.

**Example 8.** *Consider the network games* $(N,v)$ *and* $(N,v')$ *on* $N = 1,2,3$. *The first value function* $v$ *is* $v(\{12\}) = v(\{23\}) = v(\{12,23\}) = 1$, *while* $v(g) = 0$ *for all other networks. The second value function* $v'$ *is such that any nonempty network generate the same value of 1. This example can be found in Jackson [7] to criticize the Myerson value.*

*The value function* $v$ *can be expressed in terms of unanimity coefficients and basic value functions as :*

$$v = 1v_{\{12\}} + 0v_{\{13\}} + 1v_{\{23\}} - 1v_{\{12,13\}} - 1v_{\{12,23\}} - 1v_{\{13,23\}} + 1v_{g^N}.$$

*Whereas for the value function* $v'$ *:*

$$v' = 1v_{\{12\}} + 1v_{\{13\}} + 1v_{\{23\}} - 1v_{\{12,13\}} - 1v_{\{12,23\}} - 1v_{\{13,23\}} + 1v_{g^N}.$$

*Let us now compute the position value using the expression (2.11) in the network* $g' = \{12,23\}$ *:*

$$\begin{aligned}
Y_1^{PV}(g',v) &= \frac{1\times 1}{2\times 1} + 0 + 0 - 0 - \frac{1\times 1}{2\times 2} - 0 + 0 &&= \frac{1}{4}\\
Y_2^{PV}(g',v) &= \frac{1\times 1}{2\times 1} + 0 + \frac{1\times 1}{2\times 1} - 0 - \frac{1\times 2}{2\times 2} - 0 + 0 &&= \frac{1}{2}\\
Y_3^{PV}(g',v) &= 0 + 0 + \frac{1\times 1}{2\times 1} - 0 - \frac{1\times 1}{2\times 2} - 0 + 0 &&= \frac{1}{4}.
\end{aligned}$$

*Similarly for the network game* $v'$ *:*

$$\begin{aligned}
Y_1^{PV}(g',v') &= \frac{1\times 1}{2\times 1} + 0 + 0 - 0 - \frac{1\times 1}{2\times 2} - 0 + 0 &&= \frac{1}{4}\\
Y_2^{PV}(g',v') &= \frac{1\times 1}{2\times 1} + 0 + \frac{1\times 1}{2\times 1} - 0 - \frac{1\times 2}{2\times 2} - 0 + 0 &&= \frac{1}{2}\\
Y_3^{PV}(g',v') &= 0 + 0 + \frac{1\times 1}{2\times 1} - 0 - \frac{1\times 1}{2\times 2} - 0 + 0 &&= \frac{1}{4}.
\end{aligned}$$

*The position value assigns the same allocation to the players in the network* $g' = \{12,23\}$ *regardless of which of the two value functions is in place :*

$$Y^{PV}(\{12,23\},v) = Y^{PV}(\{12,23\},v') = \left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right).$$

*Note that the allocation under the position value is different than the allocation* $\left(\frac{1}{6},\frac{2}{3},\frac{1}{6}\right)$ *under the Myerson value in both games (see Jackson [7] for the computation). Player 2 is rewarded for being central in the network considered. While it is quite desirable under the value function* $v$, *under the value function* $v'$ *player 2 is not special in any way. Under* $v'$, *player 2 is not essential to generating value.*

*The link-based Shapley value for the value function v can be calculated by using formula 5.2 :*

$$\phi_1^{Sh}(v) = \frac{1\times 1}{2\times 1} + \frac{0\times 1}{2\times 1} + 0 - \frac{1\times 2}{2\times 2} - \frac{1\times 1}{2\times 2} - \frac{1\times 1}{2\times 2} + \frac{1\times 2}{2\times 3} \qquad = -\frac{1}{6}$$
$$\phi_2^{Sh}(v) = \frac{1\times 1}{2\times 1} + 0 + \frac{1\times 1}{2\times 1} - \frac{1\times 1}{2\times 2} - \frac{1\times 2}{2\times 2} - \frac{1\times 1}{2\times 1} + \frac{1\times 2}{2\times 3} \qquad = \frac{1}{3}$$
$$\phi_3^{Sh}(v) = 0 + \frac{0\times 1}{2\times 1} + \frac{1\times 1}{2\times 1} - \frac{1\times 1}{2\times 2} - \frac{1\times 1}{2\times 2} - \frac{1\times 2}{2\times 2} + \frac{1\times 2}{2\times 3} \qquad = -\frac{1}{6}.$$

*It reflects the central position occupied by player 2. However, for the value function $v'$ which is perfectly symmetric, the link-based Shapley value is $\phi(v') = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.*

The last example shows that the link-based Shapley value takes into consideration how the value changes with respect to all subnetworks. This is due to the cooperative approach we have undertaken. In the cooperative game setting, we generally consider the grand coalition as forming, and so all other possible coalitional configurations are subsets. Here we consider that the complete network is forming, and so all possible subnetworks are taken into consideration. Despite that in a network setting, the efficient networks will generally not be the complete network, we find that it is nevertheless important to take care of the value of all alternative networks, and not a subset of them, and this is achieved in considering that the complete network is forming. This can be problematic when the complete network does not maximize the value generated, as in the last example. If we observe that an efficient network is not chosen, Jackson [7] suggests the possibility of simply adjusting allocations to be proportional to the allocation that would be obtained on an efficient network. But then, we don't see why the players don't organize themselves in an efficient network. This would only be possible if there exists some external reason preventing the formation of an efficient network. A possibility to escape this problem would be to drop the efficiency axiom in the characterization of the link-based Shapley value to study semivalues (see Dubey et al. [4]). We leave this approach for future research. Our approach remains perfectly valid if we assume a kind of superadditivity assumption in which two separate networks can realize together at least what they can achieve separately. We refer to Navarro [13] for further discussions and other allocations rules.

# 7 Conclusions

This paper is a first step into the translation of network games into associated link (TU) games. We adapt the most basic notions of cooperatives games such as the core and the Shapley value in the context of network games. Some new results about the core of simple network games are presented. If our approach reveals to be successful, other insights for network games can be reached by the transposition of other well-known notions from cooperative games such as the nucleolus, kernel, convexity issues etc. How the concepts of cooperative games translated into network games behave and are related to concepts of communication games, such as the position value or Myerson value should also be investigated by the mean of more examples. Finally, we also want to develop some solution concepts that exploit the structure of link games per se, without adapting corresponding notions of cooperative games.

# Appendix : Details for Example 3

In case the player set consists of 3 players $\{i, j, k\}$ and the network $g'$ is of the form $\{ij, jk\}$, Jackson [8] provides the expression for the Myerson value. :

$$Y_j^{MV}(g', v) = \frac{v(g')}{3} + \frac{v(\{ij\})}{6} + \frac{v(\{jk\})}{6}$$
$$Y_i^{MV}(g', v) = \frac{v(g')}{3} + \frac{v(\{ij\})}{6} - \frac{v(\{jk\})}{3}$$
$$Y_k^{MV}(g', v) = \frac{v(g')}{3} - \frac{v(\{ij\})}{3} + \frac{v(\{jk\})}{6}$$

which gives in the case of example 1 :

$$Y_3^{MV}(g', v) = \frac{16}{3} + \frac{10}{6} + 0 = 7$$
$$Y_1^{MV}(g', v) = \frac{16}{3} + \frac{10}{6} - 0 = 7$$
$$Y_2^{MV}(g', v) = \frac{16)}{3} - \frac{10}{3} + 0 = 2.$$

In the case the network under consideration is $g^N$, the corresponding TU game is

$$w_{g^N}^v(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\emptyset, \{12\}, \{23\}\}; \\ 10 & \text{if } S = \{1, 3\}; \\ 19 & \text{if } S = N. \end{cases}$$

Since player 1 and 3 are symmetric with respect to this TU game, $\theta_1^{Sh}(w^v_{g^N}) = \theta_3^{Sh}(w^v_{g^N})$. We thus concentrate on the Shapley value for player 1. In the expression (2.1),

$$w^v_{g^N}(S \cup \{1\}) - w^v_{g^N}(S)$$

will only be nonzero for $S = \{3\}$ and $S = \{2, 3\}$. Hence we can compute

$$\begin{aligned}
\theta_1^{Sh}(w^v_{g^N}) &= w^v_{g^N}(\{3\} \cup \{1\}) - w^v_{g^N}(\{3\}) \left[ \frac{1!(3-1-1)!}{3!} \right] \\
&\quad + w^v_{g^N}(\{2,3\} \cup \{1\}) - w^v_{g^N}(\{2,3\}) \left[ \frac{2!(3-2-1)!}{3!} \right] \\
&= (10-0)\left[\frac{1}{6}\right] + (19-0)\left[\frac{2}{6}\right] \\
&= 8. \qquad (7.1)
\end{aligned}$$

Hence, $\theta_1^{Sh}(w^v_{g^N}) = \theta_3^{Sh}(w^v_{g^N}) = 8$ and by efficiency of the Shapley value,

$$\theta_2^{Sh}(w^v_{g^N}) = 19 - \theta_1^{Sh}(w^v_{g^N}) - \theta_3^{Sh}(w^v_{g^N}) = 3.$$

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