Quantitative versus Qualitative Growth with Recyclable Resource

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Abstract

We reassess the issue of limits to growth in an endogenous growth model of a decentralized economy where final productions require a recyclable essential material input. The model follows a material balance approach and relies on technological assumptions consistent with the material balance principle and on an explicit distinction between the material content and the quality of produced goods. Growth follows from research activities that allow firms to improve the quality of their output and to reduce the material resource intensiveness of their production process. Even though recycling is assumed perfect, we show that 1) the material balance constraint may affect the whole transitory dynamics of the growth process; 2) quantitative growth (i.e. positive growth of material output) can only be a transitory phenomenon, long term economic growth taking exclusively the form of perpetual improvements in the quality of final goods. A long term growth path is characterized by constant values of material variables (or in a less favorable scenario, by a constant negative growth rate of those variables). We establish the existence conditions of a growth path based on quality improvements and constant material variables. It may fail to exist in a decentralized framework even though it is quite feasible from a purely physical point of view.

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1 Introduction

Since the controversial book by Meadows *et al.* (2004 for the update of the 1972 edition), the debate about the physical limits to growth has remained lively. If one considers the controversies between economists, one can schematically distinguish two antagonist positions[1]. According to the first and most optimistic one (the so-called "weak sustainability" position), long run economic growth is possible within a finite world thanks to substitutions between natural resources and man-made inputs and thanks to technical progress. This position is well illustrated by the contributions of Dasgupta and Heal, Solow or Stiglitz to the Review of Economic Studies symposium on the Economics of Exhaustible Resources (1974) but many other contributions followed.

The second position is much more pessimistic about the long run growth prospects in a finite world. It first relies on a critical appraisal of the representation of the production process in neoclassical growth theory: following Georgescu-Roegen (1971), ecological economists like Cleveland and Ruth (1997) or Daly (1997) consider that neoclassical growth models rely on much too optimistic assumptions about substitution possibilities between natural and man-made inputs and about how they can be affected by technological progress. They outline in particular that neoclassical growth models ignore the physical laws (the conservation laws and the second principal of thermodynamics) that govern the transformation process of matter and energy in all human activities, in particular the production of goods and services[2]. More formally, Islam (1985) and Anderson (1987) analyze how thermodynamic laws limit the substitution elasticities between natural and man made inputs. The material balance principle constraints the asymptotic properties of production functions (i.e. their properties when man-made inputs tend to infinity). In particular, a Cobb-Douglas function of natural and man-made inputs turns out to be inconsistent with the physical laws (see also Pethig, 2006). Similarly, Baumgärtner (2004) proves that the Inada conditions for material resource inputs[3] violate the law

[1]See Ayres (2007) for a critical assessment of the two most extreme positions.

[2]Several theoretical works in an exogenous growth setting (a.o. Ayres and Miller (1980), Germain (1991), Ruth (1993)) proposed models inspired by ecological economists' criticisms and showed that physical limits to growth were then much more stringent than in a purely neoclassical setting. Two other contributions using production functions consistent with physical principles are van den Bergh and Nijkamp (1994) and Ayres and van den Bergh (2005).

[3]Most growth models that include a material resource as a production factor in an otherwise standard production function assume that the marginal productivity of the material resource approaches infinity (resp. zero) when the resource input vanishes (resp. becomes infinite).

of mass conservation because this law implies that the marginal and average products of a material resource are necessarily bounded from above. In a theoretical general equilibrium setting, Krysiak and Krysiak (2003) show that the most commonly used production functions in macroeconomic models with material resource and/or energy (including the CES function) are inconsistent with the physical laws of matter and energy conservation.

For the last twenty years, contributions to endogenous growth theory have dealt with the question of long term growth in the presence of scarce natural resources and/or pollution. But surprisingly, the vast majority of those papers (even rather recent ones) disregards the laws of physics and the ecological economists' criticisms to the neoclassical representation of the production process. For instance, Grimaud and Rougé (2003, 2005), Groth and Schou (2007) build models in which a natural resource is one of the production factors of a Cobb-Douglas technology; Stockey (1998), Hart (2004) propose growth models with pollution in which no material flow is explicitly modelled.

Other contributions to growth theory aim at taking the ecological economists' criticisms more explicitly into account. Papers like Bretschger (2005), Smulders (1995a,b, 2003), Bretschger and Smulders (2004), Pittel et al (2006) explore the long term consequences of material balance constraints and low substitution possibilities between material and man-made inputs. In spite of the resource scarcity, they all show that under some conditions, long term growth can be sustainable thanks to research and development investments. Similarly, Akao and Managi (2007) adopt a material balance approach and put forward the sustainability conditions for long term growth in an economy with finite (but recyclable) resource, pollution and bounded assimilative capacity.

To our eyes however, this last set of models does not fully take into account all the implications of the conservation laws they want to cope with. In particular, they all make explicitly or implicitly technological assumptions that ignore (partly or fully) the restrictions put forward by Anderson (op. cit) and Baumgärtner (op. cit). Rather intuitively said, the possibility of unbounded quantitative growth in an economy with scarce (but possibly renewable or recyclable) material resource relies unavoidably on the assumption that a complete dematerialization of final productions is ultimately possible. But if the material resource content of a unit of final output becomes infinitely small through time, this means that the production technology is asymptotically characterized by an infinitely large marginal product of the resource, which is inconsistent with Anderson or Baumgärtner's results. Moreover, Krysiak (2006) also outlined that the result of unlimited growth obtained in models like

Smulders (1995a,b) follows from the assumption that human capital and/or knowledge are produced without the use of matter and/or energy. However, all activities (including education and R&D) require matter and energy even when these two inputs are not embodied in the produced output. Kryziak's argument also holds for Bretschger and Smulders (2010) in the framework of a model combining a nonrenewable resource and structural change.

In this paper, we reassess the feasibility of long term growth in the framework of an endogenous growth model of a decentralized economy where final productions require an essential material input. Section 2 presents the model and its dynamics. It follows a material balance approach and relies on technological assumptions fully consistent with the material balance principle and on an explicit distinction between the material content and the quality of produced goods. Although the possible exhaustion of non-renewable resources may be an important issue for long term growth, we only consider a recyclable resource. This modelling strategy allows us to stress more obviously that physical limits to growth would exist even in an ideal world where human productions would only use renewable inputs, without causing any environmental damage. Even in such an optimistic scenario indeed, renewable resources remain finite and thus scarce. With respect to the existing literature, the use of a material balance approach in a framework where the material content and the quality of final goods are explicitly distinguished allows us to understand better the type of growth that can be feasible in the long run. Section 3 analyzes the properties and the existence conditions of a long run growth path. Given the impossibility of a complete dematerialization of man-made productions, quantitative growth (i.e. growth of material output) can only be a transitory phenomenon even though material resources can perfectly recycled. As our model shows, an everlasting quantitative growth (as standardly encountered in endogenous growth models) could only be possible in a world where the resource stock would be infinite or where productions could become totally dematerialized. Even though quantitative growth cannot last forever, a type of a "perpetual" growth based on quality improvements may however be feasible: thanks to research activities, there may exist a growth path along which material variables are constant but the quality of final goods keeps on improving. We establish the existence conditions of such a growth path and show that institutional and environmental issues interact in this respect. The way a market economy is organized matters for the existence of a decentralized balanced growth path under a material resource constraint: even when balanced growth paths based on quality improvements and constant material variables are feasible from a technological/physical point of view, individual behaviours may imply that these paths are not achievable in the decentralized economy. In our model, this may happen if the decentralized agents's saving rate is too weak (e.g. if they do not value future enough) or if they overinvest in research (at given saving rate). Limits to growth do thus not only follow from the physical constraints but also from the interactions between these constraints and the economic behaviours of decentralized agents.

Section 4 illustrates numerically the transitory dynamics of the economy and the relative importance of quantitative and qualitative growth during this transition. The growth process can be decomposed into two phases. In a first transitory phase, quantitative- and qualitative growth coexist. The relative importance of quantitative growth in total growth does not necessarily evolve monotonically through time in the early stages of the growth process but it vanishes progressively and unavoidably. The strength of quantitative growth during this transitory phase depends on the material resource constraint and the features of the ressource extraction cost function. The second phase is a balanced growth path characterised by constant values of material variables (in the most favourable scenarii), growth taking only the form of improvements in the quality of produced goods. Section 5 concludes.

2 A macro-model with freely recyclable resource

We assume two types of long-living agents: monopolistic firms and households. Firms produce final goods using a technology with two inputs, a free material ressource and productive capital. Each period, they choose a production/price policy, a research effort and an investment level that will determine their next period capital stock. Households receive the whole macroeconomic income, consume and save. There are two types of markets: the markets for monopolistic goods and a financial market on which firms borrow funds from households.

The economy faces one environmental constraint: the availability of the material resource. The evolution of its stock is the net flow of material following from production on the one hand and recycling on the other hand: each period, a part of the resource stock enters as input into the production process whereas production and consumption activities give rise to a waste of material. This waste is recycled and returns to the available material stock of the next period. Recycling is perfect and free.

In a very aggregate model like ours, the term "material" must be understood in a very broad sense and we interprete the material ressource as an aggregate of all the useful and available resources. The use of such an aggregate notion of material relies on an assumption of perfect substitution between existing resources. This assumption is certaintly too optimistic since actual substitution possibilities are finite and sometimes very weak (see Ayres, 2007) but it allows us to gain a very conservative estimate of the consequences of material constraints on (long term) growth. Our assumption of perfect recycling serves the same purpose.

2.1 Description of the monopolistic productive sector

2.1.1 The role of monopolistic competition

As we want to consider decentralized behaviours, we must assume a particular market structure. But it must be such that producers keep an individual incentive to invest in research, including in the long run. Perfect competition would not be a suitable assumption in this respect as it would imply that no producer could appropriate itself a minimal return on its research efforts when research activities only improve the quality of produced goods. Imperfect competition is thus necessary to the existence of a long run path with qualitative growth in a model like ours. The assumption of monopolistic competition as a particular form of market imperfection offers the advantage of a relative analytical simplicity. In particular, in a horizontal differentiation framework *à la* Dixit-Stiglitz (1977), it leads to well known demand functions for monopolistic goods and price behaviour of monopolistic firms.

We assume a continuum of monopolistic firms defined on $[0,1]$. A given firm $i \in [0,1]$ produces a differentiated good i and chooses its price p_{it} and its quality level q_{it} in t; it sells c_{it} units of its output to final consumers and d_{ijt} units to each other firm j, which uses it as an investment good. Following Dixit-Stiglitz, we use CES consumption and investment indices. A consumption bundle consisting of c_{it} units of each good i leads to a total consumption level given by the index

$$C_t = \left[\int_0^1 \left[\psi(q_{it}) c_{it} \right]^{\alpha} di \right]^{1/\alpha} \quad \text{with } 0 < \alpha < 1 \tag{1}$$

and where $\psi(\cdot) > 0$ is a continuous and increasing function of the quality level q_{it}. Similarly, a firm j which purchases d_{ijt} units of each good i in t builds a capital stock given by the index

$$k_{j,t+1} = \left[\int_0^1 \left[\varphi(q_{it}) d_{ijt} \right]^{\alpha} di \right]^{1/\alpha}. \tag{2}$$

$k_{j,t+1}$ represents the new capital units installed in t for $t+1$; $\varphi(\cdot) > 0$ is a continuous and increasing function of the quality level q_{it}.

At given C_t and $k_{j,t+1}$, the cost minimizing demands for good i expressed by the representative consumer and a given firm j have well-known expressions (see Appendix A): $\forall\, i$,

$$c_{it} = [\psi(q_{it})]^{\varepsilon-1} \left[\frac{p_{it}}{P_{ct}}\right]^{-\varepsilon} C_t \tag{3}$$

$$d_{ijt} = [\varphi(q_{it})]^{\varepsilon-1} \left[\frac{p_{it}}{P_{kt}}\right]^{-\varepsilon} k_{j,t+1}, \tag{4}$$

where $\varepsilon = 1/(1-\alpha)$ and the consumption and capital price indices write as follows:

$$P_{ct} = \left[\int_0^1 \left[\frac{p_{it}}{\psi(q_{it})}\right]^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}, \tag{5}$$

$$P_{kt} = \left[\int_0^1 \left[\frac{p_{it}}{\varphi(q_{it})}\right]^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}. \tag{6}$$

At given C_t and $k_{j,t+1}$, consumption and investment demands for a given good i are thus decreasing in its price (relative to the average price of the other goods) but increasing in its quality level.

In order to simplify the model presentation, we use from now on the property that the monopolistic competition equilibrium is symmetric (monopolistic firms make identical decisions)[4]: we thus omit the subscript i in the variables describing the behaviour of a typical firm or the demand for its output.

2.1.2 Technology and Material Resource

To produce y_t units of output in period t, a firm needs a quantity x_t of material resource given by:

$$x_t = [\mu_t + \chi_t]\, y_t, \tag{7}$$

where $\mu_t > 0$ is the mass of material ressource (MR hereafter) incorporated in a unit of produced good and $\chi_t > 0$ is the mass of MR wasted during the production process. These two variables describe the dependency of the technology on MR. They are affected by an endogenous technical progress that follows from an external effect linked to past research activities of all firms (see next subsection).

[4] This property holds because monopolistic goods appear symmetrically in the CES indices and because firms have access to the same technology (so that all goods also have the material content). Quite obviously, these assumptions lack of descriptive realism but introducing more heterogeneity between goods or firms would only complicate the model without changing substantially its implications. None of our qualitative results follows from the fact that the equilibrium is symmetric in the monopolistic sector.

MR is a free common resource but its extraction/transformation process requires physical capital. The capital intensiveness of this process (or the marginal transformation cost) is assumed to be increasing in the extraction rate of the resource. We denote by R_t the available aggregate stock of MR at the beginning of period t and by X_t the aggregate quantity of MR extracted/transformed during period t. The extraction rate of MR during period t, E_t, is thus $E_t = X_t/R_t \in [0, 1]$. To handle a quantity x_t of MR, a firm needs a productive capital stock ${}_pk_t$ given by

$$ {}_pk_t = x_t \Lambda(E_t) \quad \text{with} \quad \Lambda(E_t) = \zeta + \frac{\nu}{1 - E_t}, \tag{8} $$

where $\zeta \geq 0$ and $\nu > 0$ are parameters and function $\Lambda(\cdot)$ is strictly increasing in the extraction rate, the value of $\Lambda(E)$ ranging from $\Lambda(0) = \zeta + \nu > 0$ to $\Lambda(1) \to \infty$. This assumption captures the intuition that the exploitation cost of MR depends on the extent to which the available stock is used: marginal extraction costs increase when a bigger quantity is extracted and/or a lower stock remains available (see a.o. Lin et Wagner, 2007). This argument is consistent with numerous real life examples and is in line with Meadows *et al.*, (2004): for them, limits to growth due to resource scarcity should be understood more as a problem of rising costs than as one of a physical exhaustion.

If the assumption of an increasing marginal extraction cost raises little question at the level of one specific ressource, it may however seem less obvious in the case of a very aggregate material ressource like the one we consider here: if the available ressource stock is very large, one may indeed wonder whether the marginal extraction cost is increasing even at (very) low extraction rates. From a formal point of view, note first that with a parameter ν choosen sufficiently small, the marginal extraction cost may be *quasi* constant over a large interval of extraction rates. But more fundamentally, an hyperbolic function like (8) can be considered as a smooth approximation of the aggregate cost function following from the exploitation of many different ressources assumed to be perfect substitutes. Appendix B shows graphically that if the extraction cost of each ressource has rather realistically an hyperbolic form of the same type as (8), the aggregate extraction cost function will be increasing and will admit a vertical asymptote when the aggregate extraction rate approaches 1.

2.1.3 Research and Innovation

By investing in research, a monopolistic firm improves the quality of its output, which stimulates demand at given output price (see (3) and (4)). However, an innovating firm cannot appropriate itself its research results for longer than one period.

The research process and the diffusion of its results are formalized as follows. All firms enter a given period t with the same level of knowledge (or equivalently quality in the present framework), Q_{t-1}, which is the public heritage of all former research efforts. If a firm then invests in research, it raises the quality q_t of its output above Q_{t-1} and enjoys a larger output demand during the same period. The research technology is assumed to be deterministic: in order to raise quality to a level $q_t > Q_{t-1}$, a firm must endow its research department with a capital stock ${}_rk_t$ given by

$$ {}_rk_t = h\left(\frac{q_t}{Q_{t-1}}\right) y_t, \tag{9} $$

where $h(\cdot)$ is an increasing and convex function which satisfies $h(1) = 0$ (no research investment is required to maintain the quality level unchanged) and $h'(1) = 0$.

Moreover a dynamic external effect follows from individual research efforts: at the end of a period, all research results become a public good and all firms next have a free access to a quality level given by

$$ Q_t = Q(q_{it}, i \in [0, 1]) \tag{10} $$

where function Q is increasing in the individual research efforts. This level of knowledge reached at the end of t makes the production technology less material resource consuming in $t+1$:

$$ \chi_{t+1} = \chi(Q_t) \quad \text{with} \quad \chi'(\cdot) < 0 \tag{11} $$

$$ \mu_{t+1} = \mu(Q_t) \quad \text{with} \quad \mu'(\cdot) < 0. \tag{12} $$

2.1.4 Quality and material content of final productions

In our model, a final good is described by four explicit characteristics: quantity and price as usual in economics but also quality and physical content. Distinguishing the last two (endogenous) characteristics is important even though they are not necessarily unrelated: for example, the mass of a laptop can be seen as an aspect of its quality. But the quality of a production cannot be limited to its physical content (two laptops with the same mass are not necessarily equally powerful, ergonomic,...). This explicit distinction allows us to put forward a fundamental asymmetry in the way in which technological progress can change them: on the one hand, there is *a priori* no upper bound on the quality level that a production can reach thanks to a perpetual accumulation of knowledge; on the other hand, there is a lower bound on the minimal physical content of human productions. All activities (including research itself) need matter and energy. In other words, even though technological progress is *a priori* unbounded from the point of view of the quality of human productions, there is however an impossibility of a complete dematerialization of final productions and/or of production processes. This is a key presupposition of our model. From a theoretical point of view, it is fully consistent with Anderson (1987) or Baumgärtner (2004). Moreover, its descriptive realism seems undisputable: even though totally immaterial services may or might be developed, some productions will always remain partly material. Furthermore, even the production of purely immaterial services depends on material inputs and/or energy in one way or another: man-made capital inputs (such as tools, machines, vehicles, cables,...) have a minimum material content; labour input has a material content too and cannot survive without a minimum and regular material intake. Obviously, technological progress as well as sectoral reallocations of final productions towards services can increase the degree of dematerialization of aggregate output but a state of complete dematerialization of aggregate output and of its production process is only an unachievable abstraction.

Accordingly, we make the following assumptions. First, functions χ and μ are bounded from below:

$$\lim_{Q \to +\infty} \chi(Q) = \underline{\chi} > 0 \quad \text{and} \quad \lim_{Q \to +\infty} \mu(Q) = \underline{\mu} > 0. \tag{13}$$

(13) reflects the physical impossibility of a state in which a unit of final good would be produced from an infinitesimal quantity of MR. Second, productive capital cannot reach a state of complete dematerialization either: it is not possible to build an infinite capital stock level from a finite quantity of material investment good. That is, quality function $\varphi(\cdot)$ is bounded from above:

$$\lim_{q \to +\infty} \varphi(q) = \bar{\varphi} < \infty. \tag{14}$$

Hereafter, we note $\Phi(q_t)$ the (positive) elasticity of function $\varphi(\cdot)$ with respect to q_t: (14) also means that $\lim_{q \to +\infty} \Phi(q) = 0$. This assumption, which is not crucial to our results (see subsection 3.2), reflects that the production process itself cannot be completely dematerialized. The productive tools and/or the productive infrastructure have some material content; similarly, the research process at the origin of the production of knowledge cannot rely on a purely immaterial capital input.

However, the welfare impact of a rising quality level is not necessarily bounded: $\psi(\cdot)$ is thus not bounded from above. We will note $\Psi(q_t)$ the positive elasticity of function $\psi(\cdot)$ with respect to q_t. Its asymptotic value of $\Psi(q_t)$ may remain (strictly) positive: $\lim_{q_t \to \infty} \Psi(q_t) = \Psi \geq 0$.

2.1.5 Productive capital requirements

Given (7), (8) and (9), the total capital stock requirement of a monopolistic firm during period t is linked to its production and target quality levels as follows:

$$k_t = {}_pk_t + {}_rk_t = \left[(\chi_t + \mu_t)\Lambda(E_t) + h\left(\frac{q_t}{Q_{t-1}}\right)\right] y_t. \tag{15}$$

For computational simplicity, we assume a unitary depreciation rate[5]. The productive capital stock of a typical firm in a given period thus corresponds to its investment level during the previous one.

2.1.6 Price, Investment and Research Decisions

Given (3) and (4), total (consumption and investment) demand for a monopolistic good writes as

$$y_t = c_t + d_t = \psi^{\varepsilon-1}(q_t)\left[\frac{p_t}{P_{ct}}\right]^{-\varepsilon} C_t + \varphi^{\varepsilon-1}(q_t)\left[\frac{p_t}{P_{kt}}\right]^{-\varepsilon} K_{t+1}, \tag{16}$$

where $d_t = \int_0^1 d_{jt}\mathrm{d}j$ is the total investment demand addressed to a typical firm and $K_{t+1} = \int_0^1 k_{jt+1}\mathrm{d}j$ denotes the aggregate value of the capital stocks desired for $t+1$ (but installed in t).

At the beginning of period t, each firm chooses its price policy p_t and its quality level q_t; these two variables determine y_t (via (16)) and the way the existing capital stock k_t is allocated to production and research activities (via (15)). Each firm also decides on its current investment level, which will determine its next period capital stock k_{t+1}. It makes those decisions so as to maximize the following intertemporal profit function where r_t is the interest rate in t:

$$\max_{\{q_t, p_t, k_{t+1}\}_{t\geq 1}} \sum_{t=1}^{T} \frac{p_t y_t - P_{kt} k_{t+1}}{\Pi_{\tau=1}^{t}\left[1 + r_\tau\right]}$$

subject, $\forall t \geq 1$, to constraints (15), (16) and $q_t \geq Q_{t-1}$, with k_1, Q_0 and Q_{-1} given.

Let MC_t denote the marginal cost of production in t:

$$MC_t = P_{kt-1}(1 + r_t)\frac{k_t}{y_t}. \tag{17}$$

A marginal increase in output requires more (free) MR and more capital, the user cost of which is $P_{kt-1}(1 + r_t)$: productive capital in t must be purchased in $t-1$ at price P_{kt-1} and is financed by

[5] Since the productive capital stock built in t incorporates the quality of the investment goods at that time, the assumption of a depreciation rate strictly below 1 would imply that different vintages of productive capital coexist. This would complicates strongly the model without adding any substantial insight.

borrowing, which implies in t a debt service of $P_{kt-1}(1+r_t)$ per unit of capital. MC_t is the product of this user cost and the marginal capital intensiveness of output $\partial y_t/\partial k_t$, which is k_t/y_t (see (15)).

The profit maximization problem is solved in details in Appendix C and leads to the following optimality conditions for price and quality levels:

$$p_t = \frac{\varepsilon}{\varepsilon-1} MC_t \tag{18}$$

$$(p_t - MC_t)\frac{\partial y_t}{\partial q_t} = P_{kt-1}(1+r_t)h'\left(\frac{q_t}{Q_{t-1}}\right)\frac{y_t}{Q_{t-1}}. \tag{19}$$

Equation (18) shows that a monopolistic price is set by marking up the marginal cost of production. Equation (19) states that the optimal quality level must equalize the marginal benefit and cost of a quality improvement. Its left-hand-side represents the marginal income following from a marginal increase in quality: by stimulating output demand, a higher quality level raises the firm's output and operating surplus at given price. The right-hand side of (19) represents the marginal cost of this quality improvement: to increase quality, a firm needs to allocate more capital to its research department and supports the user cost of this marginal unit of capital. Multiplying (19) by q_t/y_t and using (17) and (18) allow ones to rewrite the optimality condition on q_t as follows:

$$\frac{\eta_{y \cdot q}}{\epsilon - 1}\frac{k_t}{y_t} = h'\left(\frac{q_t}{Q_{t-1}}\right)\frac{q_t}{Q_{t-1}}, \tag{20}$$

where $\eta_{y \cdot q}$ is the elasticity of demand with respect to quality. Given (16), it is a weighted average of the elasticities of functions $\psi(\cdot)$ and $\varphi(\cdot)$ with respect to q_t, i.e.

$$\eta_{y \cdot q} = (\varepsilon - 1)\left[\Psi(q_t)\frac{c_t}{y_t} + \Phi(q_t)\frac{d_t}{y_t}\right]. \tag{21}$$

The optimal investment choice in t follows from the price and research policies (18)-(19) that will be implemented in $t+1$. Obviously enough, the typical firm will make no investment in T, i.e. $k_{T+1} = 0$.

2.2 Aggregate consumption behaviour

We consider a representative and long-living agent whose behaviour is quite standard. She consumes final goods and accumulates a financial wealth Ω. In each period t, she receives the whole aggregate macroeconomic income under the form of interest rate payments $r_t\Omega_t$ and profits π_t. Her financial wealth is initially given (Ω_1 given) but next evolves as $\Omega_{t+1} = \Omega_t[1+r_t] + \pi_t - C_t, \forall t \geq 1$, where the consumption index is used as *numéraire*, i.e. P_{ct} is normalized to 1. The consumer's preferences are represented by the intertemporal utility function $\sum_{t=1}^{T} \beta^t \ln(C_t)$ where $0 < \beta < 1$ and C_t is given by (1). The optimal consumption path is well known and given by:

$$\frac{1}{C_t} \quad = \quad \beta\left[1 + r_{t+1}\right] \frac{1}{C_{t+1}}, \quad \forall t \in [1, T-1], \tag{22}$$

last period consumption satisfying the terminal condition $\Omega_{T+1} = 0$: $C_T = (1 + r_T)\Omega_T + \pi_T$.

2.3 The dynamic system

2.3.1 Relationships between price indices and monopolistic prices

In a symmetric equilibrium in the monopolistic sector, (5) becomes

$$1(= P_{ct}) = \frac{p_t}{\psi(q_t)} \qquad \text{or} \qquad p_t = \psi(q_t). \tag{23}$$

Similarly, the investment price index (6) becomes:

$$P_{kt} = \frac{p_t}{\varphi(q_t)} = \frac{\psi(q_t)}{\varphi(q_t)}. \tag{24}$$

Note that with a continuum of firms of measure 1, the distinction between K_t and k_t (and X_t and x_t) becomes virtual in a symmetric equilibrium since $K_t = \int_0^1 k_t \mathrm{d}j = k_t$ (and $X_t = \int_0^1 x_t \mathrm{d}j = x_t$).

2.3.2 Material resource stock dynamics

MR dynamics obeys the law of mass conservation: the total quantity of material in a closed system[6] is necessarily constant. Let $\mathcal{M}$ denote this constant quantity. Since flows of material residuals of a given period (following from production, consumption or capital obsolescence) are assumed to be perfectly recyclable, they enter again into the available MR stock in the following period. At the beginning of period t, MR is thus either present under the form of disposable resource R_t or embedded in the productive capital stock k_t, i.e. for all t,

$$R_t + \mu(q_{t-2}) \frac{k_t}{\varphi(q_{t-1})} = \mathcal{M}. \tag{25}$$

[6]Even though Earth is not quite a closed system, this approximation can be made for matter.

2.3.3 Summary of the dynamic system

Since firms make identical choices, condition (10) reduces to $Q_t = Q(q_t)$, and for the sake of simplicity, we simply set $Q_t = q_t$.

In each period $t \geq 1$, the macroeconomic equilibrium can be summarized by the following system of 9 equations with 9 unknowns $C_t, c_t, y_t, d_t, R_t, x_t, q_t, r_t, k_{t+1}$:

$$\frac{1}{C_t} = \beta \left[1 + r_{t+1}\right] \frac{1}{C_{t+1}} \tag{26}$$

$$y_t = c_t + d_t \tag{27}$$

$$c_t = \frac{C_t}{\psi(q_t)} \text{ and } d_t = \frac{k_{t+1}}{\varphi(q_t)} \tag{28}$$

$$\frac{k_t}{y_t} = \left[(\chi(q_{t-1}) + \mu(q_{t-1}))\Lambda\left(\frac{x_t}{R_t}\right) + h\left(\frac{q_t}{q_{t-1}}\right)\right] \tag{29}$$

$$x_t = \left[\chi(q_{t-1}) + \mu(q_{t-1})\right] y_t \tag{30}$$

$$\mathcal{M} = R_t + \mu(q_{t-2}) \frac{k_t}{\varphi(q_{t-1})} \tag{31}$$

$$\alpha \frac{\psi(q_t)}{\psi(q_{t-1})} \frac{\varphi(q_{t-1})}{1 + r_t} = \frac{k_t}{y_t} \tag{32}$$

$$\frac{k_t}{y_t}\left[\Psi(q_t)\frac{c_t}{y_t} + \Phi(q_t)\frac{d_t}{y_t}\right] = h'\left(\frac{q_t}{q_{t-1}}\right)\frac{q_t}{q_{t-1}}. \tag{33}$$

The initial conditions are k_1, q_0, q_{-1}. The terminal condition is $k_{T+1} = 0$ or $c_T = y_T$ (and $C_T = \psi(q_T)y_T$). In the sequel, we will consider that $T \to \infty$.

2.3.4 Properties of the transitional dynamics

Let $\hat{q}_t$ be the growth factor of knowledge (or quality) in t, i.e. $\hat{q}_t = q_t / q_{t-1}$.

Lemma 1

In an infinite horizon framework, the dynamics of the economy has the following properties:

1. The material consumption-output ratio (c_t/y_t) and the material investment-output ratio (d_t/y_t) are constant and respectively equal to

$$\frac{c_t}{y_t} = 1 - \alpha\beta \quad \text{and} \quad \frac{d_t}{y_t} = \alpha\beta, \ \forall t \geq 1. \tag{34}$$

Hence, y, c and d always grow at the same rate.

2. The growth factors of capital and material output write as

$$\frac{k_{t+1}}{k_t} = \frac{\alpha\beta\varphi(q_t)}{k_t/y_t} = \frac{\alpha\beta\varphi(q_t)}{(\chi(q_{t-1}) + \mu(q_{t-1}))\Lambda(E_t) + h(\hat{q}_t)} \tag{35}$$

$$\frac{y_t}{y_{t-1}} = \frac{\alpha\beta\varphi(q_{t-1})}{(\chi(q_{t-1}) + \mu(q_{t-1}))\Lambda(E_t) + h(\hat{q}_t)}. \tag{36}$$

They are increasing in the quality of investment goods (φ) and in the degree of dematerialization of physical productions (i.e. they are decreasing in χ and μ); they are decreasing in the ressource extraction rate $E_t = x_t/R_t$:

Proof: See Appendix D.

The first point of Lemma 1 follows from the choice of a logarithmic utility function. The saving rate would exhibit a transitory dynamics under a more general assumption but this would not change the nature of our results. The impacts of φ, χ and μ on capital and output growth factors are intuitive and do not need a long comment: an increase in the state of dematerialization of final productions (i.e. a decrease in χ and/or μ) and/or of the production process itself (i.e. an increase in φ)[7] enhances the productivity of MR and/or of capital and stimulates thereby output growth and capital accumulation. The negative impact of the extraction rate on output (and capital) growth can also be intuitively understood: a higher extraction rate makes the production process more capital intensive, which decreases the marginal productivity of capital and makes physical production (and thereby investment) more costly. Equations (35) and (36) show that if the extraction/transformation cost of the ressource is increasing in its utilization rate, the impact of the material balance constraint on economic growth is not solely a long term issue: it may affect (via the extraction rate) the whole transitory dynamics of material output and investment as illustrated numerically in section 4.

With the research technology we consider here, the research effort has a negative instantaneous impact on output growth but has a positive dynamic impact. On the one hand, physical production and research are rival activities as far as the use of the existing capital stock is concerned: during a given period, a bigger research effort implies less physical production. On the other hand, past research activities make the production process more efficient (via a larger φ and smaller χ and μ), which stimulates present output growth. Moreover, the instantaneous impact of research activities on capital accumulation in (35) is *a priori* ambiguous. On the one hand, the instantaneous negative output effect of research lowers physical investment. On the other hand, research improves the quality

[7] An increase of φ reduces the material content of a productive capital unit.

of investment goods, which enhances capital accumulation. Under assumption (14), the first effect dominates necessarily in the long run.

3 Long-Run Growth Paths

3.1 Properties

We define a balanced growth path (hereafter BGP) as a growth path characterized by a constant and positive growth rate of the level of knowledge (or quality) and a constant growth rate of y, c, d, k, x, R and C. Let $\hat{q}$ denote the constant growth factor of knowledge (or quality) along the BGP.

Lemma 2

Along a balanced growth path:

1. Final productions and productive capital reach their highest degree of dematerialization: $\mu(q) \overset{BGP}{\longrightarrow} \underline{\mu}, \chi(q) \overset{BGP}{\longrightarrow} \underline{\chi}$, $\varphi(q) \overset{BGP}{\longrightarrow} \bar{\varphi}$.
2. The resource input-output ratio is constant and equal to
$$\left(\frac{x_t}{y_t}\right)_{BGP} = \underline{\chi} + \underline{\mu}. \tag{37}$$
3. Material variables, i.e. y_t, c_t, d_t and x_t, have the same growth factor as productive capital k_{t+1}. It is the following decreasing function of the extraction rate $\tilde{E} = \tilde{x}/\tilde{R}$ and the growth factor of knowledge $\hat{q}$:
$$\left(\frac{y_t}{y_{t-1}}\right)_{BGP} = \left(\frac{k_{t+1}}{k_t}\right)_{BGP} = \frac{\alpha\beta\bar{\varphi}}{(\underline{\chi} + \underline{\mu})\Lambda(E_t) + h\left(\hat{q}\right)}. \tag{38}$$
4. The growth rate of material variables is either nil or negative.

Proof: See Appendix E.

We label positive (resp. negative) the balanced growth path associated to constant values (resp. a constant negative growth rate) of material variables. Proposition 1 hereafter analyzes the properties of a positive BGP (in short, PBGP). Section 3.3 will discuss its feasibility and conditions of existence.

For notational convenience, let us define function $H(\hat{q})$ as $H(\hat{q}) = h'(\hat{q}) \cdot \hat{q}$. Note that $H(1) = h'(1) = 0$ and $H'(\hat{q}) = h''(\hat{q})\hat{q} + h'(\hat{q}) > 0$ since $h(\cdot)$ is an increasing and convex function.

Proposition 1

In the presence of a limited but perfectly recyclable essential material resource, quantitative output growth can only be a transitory phenomenon: perpetual economic growth can only take the form of perpetual improvements in the quality of final goods. That is, along a PBGP of the decentralized economy:

1. *Material variables y, c, d, R, x and the productive capital stock k are constant.*

2. *The growth factor of knowledge is*

$$\hat{q} = H^{-1}\left(\bar{\varphi}\alpha\beta(1-\alpha\beta)\Psi\right) \tag{39}$$

and implies perpetual improvements in the quality of consumption goods, which are thus the only source of long term growth of the consumption index C_t and the representative agent's welfare.

3. *The extraction rate of the resource is*

$$\tilde{E} = 1 - \frac{(\underline{\chi}+\underline{\mu})\nu}{\bar{\varphi}\alpha\beta - (\underline{\chi}+\underline{\mu})\zeta - h(\hat{q})}. \tag{40}$$

4. *Material variables are (linear) functions of the material resource endowment:*

$$\tilde{R} = \Sigma(\tilde{E})\mathcal{M} \quad \textit{with} \quad 0 < \Sigma(\tilde{E}) = \frac{\underline{\chi}+\underline{\mu}}{\underline{\chi}+\underline{\mu}+\alpha\beta\underline{\mu}\tilde{E}} \leq 1 \tag{41}$$

$$\tilde{x} = \tilde{E}\Sigma(\tilde{E})\mathcal{M} \tag{42}$$

$$\tilde{y} = \tilde{E}\Sigma(\tilde{E})\frac{\mathcal{M}}{\underline{\chi}+\underline{\mu}} < \frac{\mathcal{M}}{\underline{\chi}+\underline{\mu}} \tag{43}$$

and $\tilde{d} = \alpha\beta\tilde{y}$, $\tilde{k} = \bar{\varphi}\alpha\beta\tilde{y}$, $\tilde{c} = (1-\alpha\beta)\tilde{y}$.

Proof: See Appendix F.1.

Intuitively enough, the growth of knowledge implied by (39) (and thereby the growth rate of the consumption index) is an increasing function of Ψ and $\bar{\varphi}$. A high elasticity of the quality of consumption goods with respect to research effort (a high Ψ) is indeed an incentive to invest in research. Similarly, the more productive/research capital can be dematerialized (the higher $\bar{\varphi}$), the less costly research is (in terms of foregone physical production): the incentive to invest in it is then stronger.

The growth rate of q is however a non monotonic function of the saving rate $\alpha\beta$: it is increasing if $\alpha\beta < 1/2$ and decreasing otherwise[8]. On the one hand, a higher saving rate implies a stronger capital

[8] The partial derivative of $\alpha\beta\bar{\varphi}(1-\alpha\beta)\Psi$ with respect to $\alpha\beta$ is equal to $\bar{\varphi}(1-2\alpha\beta)\Psi$ and is positive iff $\alpha\beta < 1/2$.

accumulation and a higher capital/output ratio, which stimulates research effort *ceteris paribus*. On the other hand, along a BGP and under assumption (14), research only improves the quality of consumption goods. From this perspective, the lower the output share allocated to consumption, the weaker the long run incentive for research. When the consumption/output ratio is low enough (or $\alpha\beta$ high enough), this second effect dominates.

The determinants of the long run extration rate (40) play a quite intuitive role. First, the larger the marginal cost of extraction (i.e. the larger ν or the larger ζ), the lower the extraction rate. At given ν and ζ, a higher degree of dematerialization of final goods (i.e. a lower $\underline{\chi}$ or $\underline{\nu}$) or of the production process (i.e. a higher $\bar{\varphi}$) eases the accumulation of productive capital which allows the economy to sustain a higher extraction rate. Finally, the negative relationship between $\tilde{E}$ and $\hat{q}$ reflects that research and physical production are rival activities as far as the use of the (stationary) capital stock is concerned: the higher the research effort, the lower the capital stock available for production activities and the lower the extraction rate.

As (43) shows, the stationary material output level (and the related material variables) will be increasing in the resource stock[9] and its extraction rate. Moreover, the more final productions can be dematerialized (i.e. the lower $\underline{\chi}$ or $\underline{\nu}$), the larger the stationnary output level will be.

3.2 About limit cases with unbounded quantitative growth

Our model combines an endogenous growth model with core assumptions of ecological economics. We identify here under what assumptions it would reproduce usual endogenous growth results.

Proposition 2

> Perpetual growth of material output y would occur if any one of the two following conditions was met:
>
> 1. The available quantity of material resource is unlimited, i.e., $\mathcal{M} \to \infty$.
> 2. In the long run, technological progress allows firms to produce final goods characterized by a complete dematerialization: i.e. $\underline{\mu} \to 0$ and $\underline{\chi} \to 0$.

[9] The linearity of $\tilde{y}$ with respect to $\mathcal{M}$ follows from our technological assumption and the fact that $\hat{q}$ and $\tilde{E}$ do not depend on $\mathcal{M}$: as y depends linearly on x, it is linear in $\mathcal{M}$ too.

Proof: See Appendix F.2.

Case 1 is impossible in a finite world; Case 2 would mean that technological progress could ultimately free the production processes from physical laws. In order to comment this proposition further, let us rewrite (15) as follows: $y_t = \left[(\mu_t + \chi_t)\Lambda(E_t) + h\left(\hat{q}_t\right)\right]^{-1} k_t$. In our model, final output depends linearly on capital with an endogenous productivity of capital. Along a balanced growth path where $E_t = \tilde{E}$, $\hat{q}_t = \hat{q}$, $\mu_t = \underline{\mu}$ and $\chi_t = \underline{\chi}$, the productivity of capital in the above expression reaches a constant value and the BGP technology of our model is of the AK-type: $y_t = \text{constant} \times k_t$: our model would thus behave as a fairly standard endogenous growth model if the ressource constraint was not playing an active role. As stated in the first point of proposition 2, it would be certainly the case if there was no binding ressource constraint in the model (i.e. if $\mathcal{M} \to \infty$): this corresponds to the assumption implicitly made in all (endogenous) growth models without any material ressource constraint. But beyond this first and obvious case, there is a more subtle one in which the ressource constraint (albeit present) would not actually be binding in the long run: even with a finite ressource level, perpetual growth of material output would still be possible if productions could become totally dematerialized. Indeed, it would then be possible to produce infinite capital and output levels from a finite material stock. The second limit case in proposition 2 puts forward the hypothesis that is made more or less implicitly (but unavoidably) in all growth models that incorporate a material resource constraint but nevertheless get a result of everlasting growth of material output. In some papers, this hypothesis is consubstantial with the type of technology assumed for the productive sector (a Cobb Doublas for instance, see our introduction). But in other models, it is implicitly present in the assumptions characterizing the technology of the research sector: the research sector is then typically assumed not to use any material resource input directly or indirectly (i.e. via a man-made capital that would have some material content). This is a.o. the case in Bretschger and Smulders (2010) who build a model combining nonrenewable resources and structural change. The technology of their productive sectors contains a productivity index that can grow without limit thanks to the diffusion of a technological progress which is the output of a research sector that does not use any (form of) material input.

Note that the impossibility of long run growth of material output (under $\mathcal{M} < \infty$ and (13)) does not depend on assumption (14) on φ. Even though capital input could reach a state of complete dematerialization, material output growth would remain purely transitory: with $\bar{\varphi} \to \infty$, an infinite capital stock could be asymptotically built from an infinitely small quantity of material good (the extraction could even tend to 1) and all material productions could be progressively allocated to consumption; however, material output would remain bounded by $\mathcal{M}/[\underline{\chi}+\underline{\mu}]$ (see (43).

3.3 Feasibility of a PBGP and existence of a decentralized PBGP

Before determining the existence conditions of a decentralized PBGP, we analyze the feasibility conditions of a PBGP from a purely technological/physical viewpoint. A PBGP will be feasible if it is such that the saving rate and the extraction rate belong to $[0,1]$ ($0 \leq d/y, \tilde{E} \leq 1$) and there is a positive research effort, i.e. $1 \leq \widehat{q}$ (or $0 \leq h(\widehat{q})$). In order to determine the existence conditions of feasible PBGPs, we set aside the model equations describing the decentralized agents' decisions and we focus on technological and physical constraints (29), (30), (31) and accounting identities (27), (28). Along a PBGP, this set of equations can easily be reduced to the following relationship:

$$\overline{\varphi}\frac{d}{y} = h(\widehat{q}) + (\underline{\chi}+\underline{\mu})\Lambda(\tilde{E}). \tag{44}$$

At given d/y, (44) defines a negative relationship between $\tilde{E}$ and $\hat{q}$: with a higher extraction rate, the capital requirement per unit of output, $(\underline{\chi}+\underline{\mu})\Lambda(\tilde{E})$, increases and the share of the capital stock allocated to research is thus lower.

Proposition 3

1. A feasible PBGP can be associated to any couple $(\tilde{E}, \hat{q})$ such that

$$0 < \tilde{E} \leq 1 - \frac{(\underline{\chi}+\underline{\mu})\nu}{\overline{\varphi} - (\underline{\chi}+\underline{\mu})\zeta} \tag{45}$$

$$0 < h(\widehat{q}) < \overline{\varphi} - (\underline{\chi}+\underline{\mu})\Lambda(\tilde{E}). \tag{46}$$

2. A necessary condition for a PBGP to be feasible is

$$\overline{\varphi} > (\underline{\chi}+\underline{\mu})\Lambda(0) = (\underline{\chi}+\underline{\mu})(\zeta+\nu). \tag{47}$$

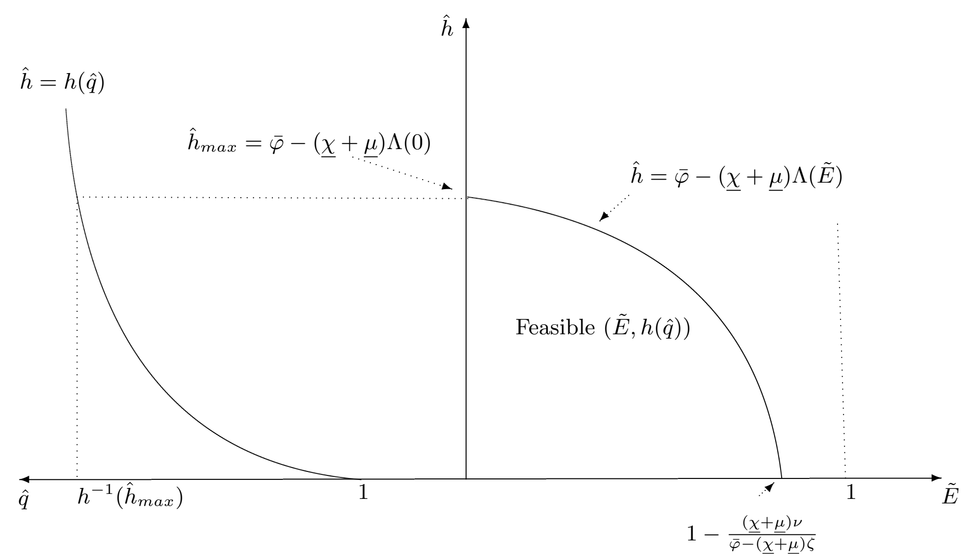
Proof: See Appendix E.3

The upper bound of $\tilde{E}$ in (45) shows that the ressource extraction rate cannot be too high along a feasible PBGP. With an increasing extraction cost (i.e. with $\nu > 0$), this upper bound is indeed strictly smaller than one if a complete dematerialization of final productions and/or of the production process is impossible (i.e. under assumptions (13) and (14)). A too high extraction rate (i.e. above the upper bound of (45)) would require a capital stock level that the economy is physically unable to sustain on a stationary basis. Intuitively enough, the highest $\tilde{E}$ consistent with the existence of a feasible PBGP is decreasing in ν but increasing in the degree of dematerialization of output and/or of its production process (it is thus decreasing in $\underline{\chi}$ and $\underline{\mu}$ but increasing in $\bar{\varphi}$).

Similarly, the upper bound of $\tilde{E}$ in (46) shows that the reseach effort cannot be too strong along a feasible PBGP. The (stationnary) capital stock is allocated either to production or to research activities. At given extraction rate, if the research effort was above the upper bound of (46), too little capital would be allocated to production activities and the economy would be unable to keep its capital stock constant.

Figure 1 illustrates Proposition 3.

Figure 1: Domain of feasibility of the PBGPs



With a unitary saving rate, (44) corresponds to curve $h(\widehat{q})$ in the right panel of Figure 1 and is the frontier of the set of feasible values of $\tilde{E}$ and $h(\hat{q})$: this set consists of all couples $(\tilde{E}, \widehat{h})$ of the positive orthant under this curve. A fully informed central planner could choose one of these couples (and the corresponding saving rate) in order to maximize a chosen social objective. If (47) does not hold, the curve intercept is negative and no PBGP is feasible: the efficiency of material investment good would be too low and the economy would be unable to sustain a constant capital stock level even in the scenario which is the most favourable to capital accumulation, i.e. a scenario in which 1) the stock would be exclusively allocated to productive activities (no research investment), 2) the extraction rate would be nil and 3) final production would only be allocated to investment. Assume indeed that a given $\tilde{k}$ could be a stationary state capital stock level. In the scenario just described ($\hat{q} = 1$, $\tilde{E} = 0$, $\tilde{d}/\tilde{y} = 1$), output (and thus investment) would be $\tilde{y} = \tilde{d} = \tilde{k}/((\underline{\chi} + \underline{\mu})\Lambda(0))$. The capital stock of the following period would be $\bar{\varphi}\tilde{d} = \tilde{k} \cdot \bar{\varphi}(\underline{\chi} + \underline{\mu})\Lambda(0)) < \tilde{k}$ if (47) does not hold: the capital stock would decrease through time.

We now analyze the existence conditions of a decentralized PGBP.

Proposition 4

1. *A decentralized PBGP exists if and only if*

$$(h(\widehat{q}) =)\, h\left(H^{-1}\left(\bar{\varphi}\alpha\beta(1-\alpha\beta)\Psi\right)\right) < \overline{\varphi}\alpha\beta - (\underline{\chi} + \underline{\mu})\Lambda(0) \qquad (48)$$

2. *A necessary condition for a decentralized PBGP to exist is a sufficiently high saving rate, i.e.*

$$\alpha\beta > \frac{\underline{\chi} + \underline{\mu}}{\overline{\varphi}}\Lambda(0). \qquad (49)$$

Proof: See Appendix E.4.

The intuition behind conditions (48) and (49) is the following one.

With respect to (46), condition (48) puts forward that an overinvestment in research is possible in the decentralized economy. Consider Figure 1 again: if the decentralized research effort is such that (48) does not hold, the value of $h(\hat{q})$ in the decentralized economy is above the intercept of the curve delimitating the frontier of the set of feasible PBGP's.

But (49) shows that a decentralized PBGP may also fail to exist if the decentralized saving rate is too low (or if the efficiency of material investment goods is too low at given saving rate). (49) is equivalent to inequality (48) in the case where $h(\hat{q})$ is nil. The intuition behind condition (49) is quite similar to the one that underlies (47) except that the saving rate is set to one in the latter and equal to its decentralized equilibrium value in the former: a negative BGP would be unavoidable if the economy was unable to maintain its capital stock constant even in the scenario which is the most favourable to capital accumulation. In the decentralized economy, this would be the case if the saving rate was too low to maintain the capital stock constant even when no investment in research is made and the extraction rate is nil. Given that the expression of k_{t+1}/k_t in (35) is decreasing in the extraction rate, the following inequality must hold along a PBGP:

$$\left(\frac{k_{t+1}}{k_t}\right)_{\text{BGP}} = \frac{\bar{\varphi}\alpha\beta}{(\underline{\chi}+\underline{\mu})\Lambda(\tilde{E})+h(\hat{q})} = 1 < \frac{\bar{\varphi}\alpha\beta}{(\underline{\chi}+\underline{\mu})\Lambda(0)+h(\hat{q})}.$$

If (48) did not hold, the upper bound of this inequality would be lower than 1 (even when no research is made), which would be mean that a PBGP cannot exist.

The comparison between the necessary conditions (47) and (49) makes clear that a decentralized PBGP may not exist even though the domain of feasible PBGPs is not empty. If the saving rate in the decentralized framework is such that

$$\frac{(\underline{\chi}+\underline{\mu})}{\alpha\beta}\Lambda(0) > \bar{\varphi} > (\underline{\chi}+\underline{\mu})\Lambda(0),$$

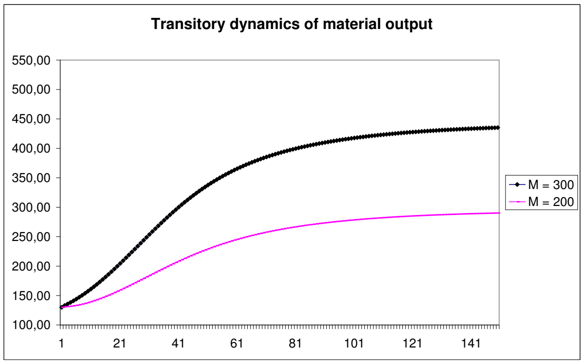
there is no decentralized PBGP although there exist feasible PBGPs. The likelihood of such a situation is higher the more decentralized agents are impatient (the lower β) and/or the stronger the market imperfections (the lower α). Given our first comment on Proposition 4, the existence of a decentralized PGBP does not only require a sufficiently high saving rate: a sufficiently large part of the saving must also be allocated to the accumulation of productive capital.

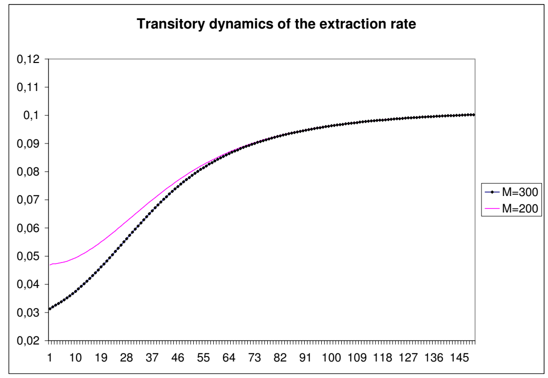
4 Transitional dynamics: a numerical exploration

A numerical illustration is necessary to analyze the transitional dynamics further than in Lemma 1. The model calibration is detailed in Appendix G. In order to illustrate the role of the resource stock, we compare numerically the transitory dynamics of two economies that differ only in their resource endowment ($\mathcal{M} = 200$ in one economy and 300 in the other). These economies start from the same initial condition in which the capital stock is below its PBGP level. Figure 2 hereafter illustrates the evolution of material output y_t and extraction rate E_t when $\zeta = 0$ and $\nu = 1$.

Beyond the scale effect of the resource stock on the material output levels of the two economies, two points are worth being outlined. First, limits to quantitative output growth manifest themselves even

Figure 2: Transitory dynamics of 2 economies with different resource endowments

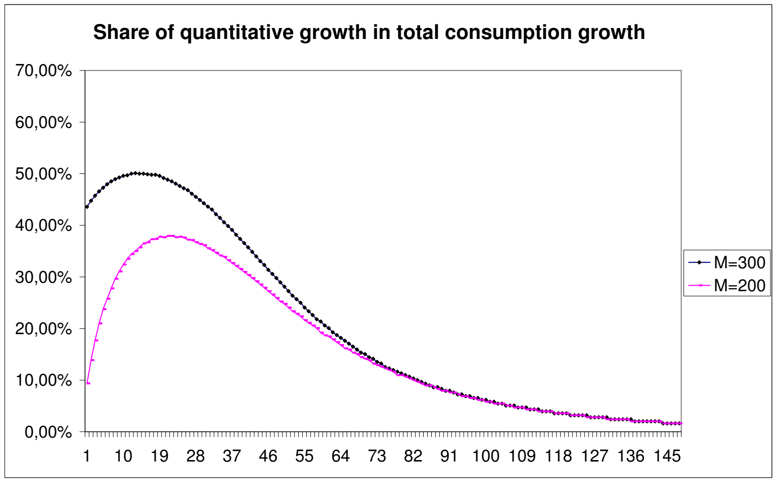




though the two economies only use a small fraction of the available resource stock (E_t is well below 1). Second, the length of the transitory phase with quantitative growth is about the same in the two economies: in our model, quantitative growth is thus stronger in the better endowed economy but it is not more lasting. In their debates with ecological economists, orthodox growth economists are used to say that a growth model that ignores environmental constraints might still provide an acceptable representation of the growth process in the short or medium run (see e.g. Stiglitz, 1997). Our simulation exercise questions this claim. Environmental constraints do not only matter in the long run: if $\Lambda(E)$ is increasing even at relatively low extraction rates, the ressource constraint can affect much earlier phases of the growth process. Some argue that such constraint do not seem to have constrained growth very severely in the Western world during the 19th and 20th centuries. But our analysis illustrates that economic growth has not occurred out of the environmental constraints: had the available environmental resources been different, so would probably have been the growth process.

As we model quality explicitly, we can break down the growth rate of output or consumption into its quantitative and qualitative components as in Figure 3. It displays the contribution of quantitative growth to total consumption growth ($C_t = q_t^{\eta} \cdot c_t$ with the chosen calibration). During the transitory dynamics, quantitative and qualitative growth coexist. Material growth may even play the dominant role during the early phase of the growth process. But it vanishes progressively.

Figure 3: Share of quantitative growth in total consumption growth



5 Concluding remark

In this paper, we have introduced a non-trivial material ressource constraint in an otherwise "standard" endogenous growth model and we have reassessed the feasibility of long term growth in a finite world. We have designed our modelling strategy in a way that aims at narrowing the gap between orthodox growth theory and ecological economics. In this regard, the main originality of our modelling is not to follow a material balance approach[10] but to make technological assumptions totally consistent with the material balance principle in a framework where the quality and the material content of final goods are explicitly distinguished. A rather thorough summary of our results has been given in the second part of the introduction and will not be repeated here. Let us only recall that if the ressource constraint affects the type of economic growth that can be possible in the long run, it also influences the transitory dynamics of a growing economy. These shorter-term implications of the ressource constraint manifest themselves all the more quickly when the marginal cost of the extraction/transformation process of the ressource is rapidly increasing in its utilization rate.

Even though our model describes quantitative growth as a transitory phenomenon, it is worth stressing that it relies on very optimistic assumptions: all existing resources have been assumed to be perfect substitutes; energy has been ignored; recycling has been considered free and perfect (so that

[10] As mentionned in the introduction, other endogenous growth models do it.

the material resource is *de facto* renewable); we have neglected the consequences of the environmental damage linked to production/consumption activities and the possibility of thresholds in the assimilative capacity of the environment. In Fagnart-Germain (2009), we have introduced renewable energy as another essential input. This extension does not affect the nature of our results but makes the existence conditions of a balanced growth more restrictive. Similarly, less optimistic assumptions about substitution possibilities between different materials, about recycling possibilities or about the assimilative capacity of the environment are likely to affect the features of a balanced growth path (and the transitory dynamics towards it) and to constraint further its existence conditions. Such topics are left for future research.

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Appendix A: Demand for monopolistic goods

We derive here the consumption demands for monopolistic goods (the derivation of the investment demands is quite similar and can be easily done *mutatis mutandis*). For a given level of the consumption index, C_t, the consumer chooses her consumption bundle so as to minimize its cost, i.e. she solves $\min_{c_{it}} \int_0^1 p_{it}c_{it}di$ subject to the equality constraint (1). Let $\lambda_t > 0$ be the Lagrange multiplier associated to constraint (1). The Lagrange function associated to this minimization problem is:

$$L_t = \int_0^1 p_{it}c_{it}di + \lambda_t \left[C_t - \left[\int_0^1 \left(\psi(q_{it})c_{it}\right)^{\alpha} \mathrm{d}i \right]^{1/\alpha} \right]$$

The first order condition on a given good i writes as follows:

$$p_{it} = \lambda_t \frac{\partial C(\cdot)}{\partial c_{it}}$$

where we use the notation $C(\cdot)$ to represent the function at the right-and-side of (1). Multiplying each FOC by c_{it} and summing all these FOCs over i gives :

$$\int_0^1 p_{it} c_{it} \mathrm{d}i = \lambda_t \int_0^1 \frac{\partial C(\cdot)}{\partial c_{it}} c_{it} \mathrm{d}i = \lambda C_t,$$

the last equality following from Euler theorem for homogenous functions of degree 1. From this later equality, it follows that λ is nothing but the price index P_{ct} associated to the consumption bundle with P_{ct} such that $\int_0^1 p_{it} c_{it} \mathrm{d}i = P_{ct} C_t$. Each optimality condition can thus be rewritten as:

$$\frac{p_{it}}{P_{ct}} = \underbrace{\left[\int_0^1 (\psi(q_{it}) c_{it})^{\alpha} \, \mathrm{d}i \right]^{\frac{1}{\alpha}-1}}_{C_t^{1-\alpha}} (\psi(q_{it}))^{\alpha} c_{it}^{\alpha-1}.$$

Therefore,

$$\frac{c_{it}}{C_t} = (\psi(q_{it}))^{\frac{\alpha}{1-\alpha}} \left(\frac{p_{it}}{P_{ct}} \right)^{-\frac{1}{1-\alpha}}, \quad \text{which is equivalent to (3).}$$

Inserting this expression of c_{it} into $P_{ct} C_t = \int_0^1 p_{it} c_{it} \mathrm{d}i$ gives the consumption price index:

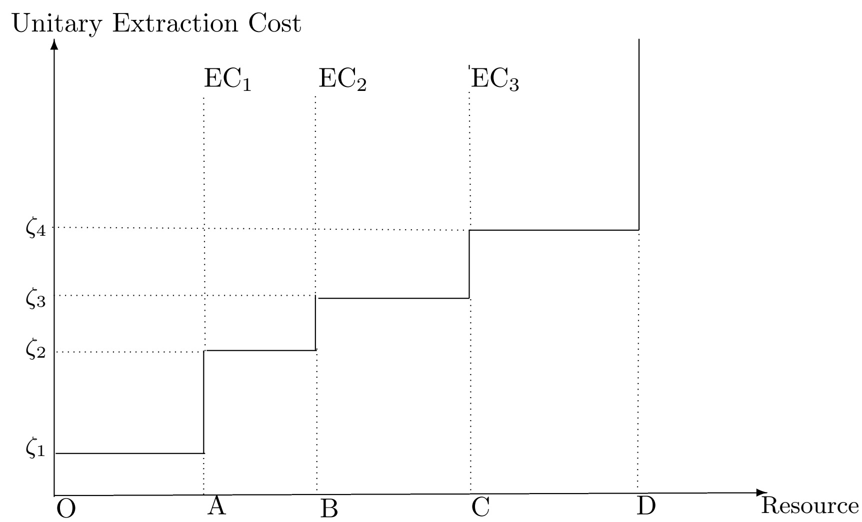
$$\begin{aligned} P_{ct} C_t &= \int_0^1 p_{it} \left[\psi(q_{it}) \right]^{\varepsilon-1} \left[\frac{p_{it}}{P_{ct}} \right]^{-\varepsilon} C_t \mathrm{d}i \\ P_{ct} &= \int_0^1 p_{it} \left[\psi(q_{it}) \right]^{\varepsilon-1} \left[\frac{p_{it}}{P_{ct}} \right]^{-\varepsilon} \mathrm{d}i \\ P_{ct}^{1-\varepsilon} &= \int_0^1 \left[\psi(q_{it}) \right]^{\varepsilon-1} (p_{it})^{1-\varepsilon} \mathrm{d}i, \qquad \text{which is equivalent to (5).} \end{aligned}$$

Appendix B: Rationalizing an increasing extraction cost function

Assume there are N ressources $i = 1, ..., N$ and that the available stock of ressource i is R_i. The extraction cost of each ressource is given by $\Lambda_i(E_i)x_i$ where x_i is the quantity of ressource i used in production and E_i is the extraction rate of ressource i; $\Lambda_i(E_i) = \zeta_i + \nu_i/(1 - E_i)$. For the simplicity of the graphical illustration, consider that $N = 4$ and that all the ressources have a different value of ζ_i but are such that $\nu_i \to 0, \forall i$: each function $\Lambda_i(E_i)$ is inversed-L shaped. Ressource 1 is available in quantity 0A, ressource 2 in quantity AB; the stocks of resources 3 and 4 are respectively equal to BC and CD. The aggregate ressource stock is OD (since all ressources are perfect substitutes)

The first ressource to be used will be one with the smallest ζ_i (ressource 1 in our example). When ressource 1 is fully used (its extraction rate tends to 1), the aggregate extraction rate is only OA/OD.

If more resource is needed, the second cheapest ressource is then used, and so on. The aggregate extraction cost function corresponds to the 4-step continuous line in the picture. It thus appears globally increasing even though the extraction cost of each ressource is constant for any extraction rate strictly below 1. Our technological assumption (8) can be interpreted as a continuous approximation of such an aggregate cost function when the number N of ressources is large enough.



An equivalent argument can be made in the more general case where the parameters ν_i's are not *quasi* nil. The extraction cost curve of each ressource has an hyperbolic shape (instead of an inversed-L shape). Using the same graphical construction as in the above case, the aggregate cost function would be a continuously increasing function with kinks (one kink each time a new ressource is engaged) and an asymptote at the level of a unitary extraction rate of the aggregate ressource. The larger the number of ressources, the better the quality of the approximation offered by the hyperbolic function (8).

Appendix C: Monopolistic Firms' Behaviour

Using (15) and (16) to express k_t and y_t as functions of q_t and p_t and associating multiplier ν_t to constraint $q_t \geq Q_{t-1}$, we write the Lagrangean of the profit maximization problem as follows:

$$\mathcal{L} = \sum_{t=1}^{T} \rho_t \left\{ p_t y_t - P_{kt} \left((\chi(Q_t) + \mu(Q_t))(\Lambda(E_{t+1}) + h\left(\frac{q_{t+1}}{Q_t}\right)\right) y_{t+1} + \nu_t \cdot (q_t - Q_{t-1}) \right\}, \rho_t = \frac{1}{\Pi_{\tau=1}^{t}(1+r_\tau)}.$$

It admits the following first order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_t} &= \rho_{t-1}\left[-P_{k,t-1}\frac{k_t}{y_t}\frac{\partial y_t}{\partial p_t}\right] + \rho_t\left[y_t + p_t\frac{\partial y_t}{\partial p_t}\right] \leq 0, ... \\
& \quad p_t \geq 0 \quad \text{and} \quad p_t\frac{\partial \mathcal{L}}{\partial p_t} = 0, \ \forall t \geq 1; \qquad (50) \\
\frac{\partial \mathcal{L}}{\partial q_t} &= -\rho_{t-1}P_{k,t-1}\left[\frac{k_t}{y_t}\frac{\partial y_t}{\partial q_t} + h'\left(\frac{q_t}{Q_{t-1}}\right)\frac{y_t}{Q_{t-1}}\right] + \rho_t\left[p_t\frac{\partial y_t}{\partial q_t} + \nu_t\right] \leq 0, ... \\
& \quad q_t \geq 0 \quad \text{and} \quad q_t\frac{\partial \mathcal{L}}{\partial q_t} = 0, \ \forall t \geq 1; \qquad (51) \\
\frac{\partial \mathcal{L}}{\partial \nu_t} &= q_t - Q_{t-1} \geq 0, \ \nu_t \geq 0 \quad \text{and} \quad \nu_t\left[q_t - Q_{t-1}\right] = 0, \ \forall t \geq 1. \qquad (52)
\end{aligned}$$

Obviously enough, $p_t = 0$ cannot be a profit maximizing choice. Hence $\partial\mathcal{L}/\partial p_t = 0$. Similarly, (52) and the initial condition $Q_0 > 0$ imply that $q_t \geq Q_{t-1} \geq Q_0 > 0$ for all t, so that $\partial\mathcal{L}/\partial q_t = 0$. After simplifying the discount factors in (50) and (51) and multiplying (50) (resp. (51)) by p_t/y_t (resp. q_t/y_t), we rewrite the above system as follows: $\forall t \geq 1$,

$$\begin{aligned}
p_t\left(1 + \frac{p_t}{y_t}\frac{\partial y_t}{\partial p_t}\right) &= (1+r_t)P_{k,t-1}\frac{k_t}{y_t}\left(\frac{p_t}{y_t}\frac{\partial y_t}{\partial p_t}\right); \qquad (53) \\
p_t\frac{q_t}{y_t}\frac{\partial y_t}{\partial q_t} + \nu_t\frac{q_t}{y_t} &= (1+r_t)P_{k,t-1}\left[\frac{k_t}{y_t}\frac{q_t}{y_t}\frac{\partial y_t}{\partial q_t} + h'\left(\frac{q_t}{Q_{t-1}}\right)\frac{q_t}{Q_{t-1}}\right] \qquad (54) \\
q_t - Q_{t-1} &\geq 0; \ \nu_t \geq 0 \quad \text{and} \quad \nu_t\left[q_t - Q_{t-1}\right] = 0, \qquad (55)
\end{aligned}$$

where the unknowns are q_t, p_t, ν_t for $t \geq 1$ and k_t for $t \geq 2$.

The price elasticity of demand q_t is $-\epsilon$ and (53) becomes equation (18) in the main text. A few simple algebraic manipulations allow ones to write (54) as follows:

$$\left[p_t - P_{k,t-1}(1+r_t)\frac{k_t}{y_t}\right] \cdot \eta_{y \cdot q} + \nu_t\frac{q_t}{y_t} = (1+r_t)P_{k,t-1}h'\left(\frac{q_t}{Q_{t-1}}\right)\frac{q_t}{Q_{t-1}} \qquad (56)$$

where $\eta_{y \cdot q} > 0$ is the elasticity of y_t with respect to q_t, i.e.,

$$\begin{aligned}
\frac{q_t}{y_t}\frac{\partial y_t}{\partial q_t} &= \frac{q_t}{y_t}\left(\frac{\partial c_t}{\partial q_t} + \frac{\partial d_t}{\partial q_t}\right) \\
&= [\varepsilon - 1]\frac{q_t}{y_t}\left(\psi^{\varepsilon-2}(q_t)\psi'(q_t)\left[\frac{P_{ct}}{p_t}\right]^{\varepsilon} C_t + \varphi^{\varepsilon-2}(q_t)\varphi'(q_t)\left[\frac{P_{k,t-1}}{p_t}\right]^{\varepsilon} K_{t+1}\right) \\
&= [\varepsilon - 1]\left[q_t\frac{\psi'(q_t)}{\psi(q_t)}\frac{c_t}{y_t} + q_t\frac{\varphi'(q_t)}{\varphi(q_t)}\frac{d_t}{y_t}\right].
\end{aligned}$$

Given equation (56), assumption $h'(1) = 0$ is a sufficient condition for an interior solution for q_t , i.e. an optimal value of $q_t > Q_{t-1}$ for all $t \geq 1$. Indeed assume $\nu_t > 0$: (55) then implies $q_t = Q_{t-1}$ (or $q_t/Q_{t-1} = 1$) and the right and side of (56) becomes nil. Given the optimal price behaviour implies, the first term at the left hand side of (56) is strictly positive and a nil value of the whole left hand side would thus require that $\nu_t < 0$, which is a contradiction.

Using $\nu_t = 0$ and the optimal price behaviour, (56) writes as follows:

$$\frac{p_t}{\epsilon} \cdot \eta_{y \cdot q} = \frac{\epsilon - 1}{\epsilon} p_t \frac{y_t}{k_t} h' \left(\frac{q_t}{Q_{t-1}} \right) \frac{q_t}{Q_{t-1}}, \quad \text{which is equivalent to (20).}$$

Appendix D: Proof of Lemma 1

1. Using (28), the optimality condition on consumption (26) can be rewritten as

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1}) \frac{\psi(q_t)}{\psi(q_{t+1})}. \tag{57}$$

   Using (57) one-period lagged, the optimal pricing rule (32) becomes

$$\begin{aligned} \alpha\varphi(q_{t-1}) \frac{y_t}{k_t} &= \frac{\psi(q_{t-1})}{\psi(q_t)} (1 + r_t) \\ &= \frac{1}{\beta} \frac{c_t}{c_{t-1}}. \end{aligned} \tag{58}$$

   After substituting k_t by its value in (28) and using (27), (58) becomes:

$$\alpha\beta \frac{y_t}{c_t} = \frac{y_{t-1}}{c_{t-1}} - 1.$$

   Let z_t be the inverse of the consumption-output ratio, i.e. $z_t = y_t/c_t$. The last equation above is a first-order linear autonomous equation in z: $z_{t-1} = \alpha\beta z_t + 1$. A particular solution of this equation is the constant value $\tilde{z} = (1 - \alpha\beta)^{-1}$. Its general solution is $z_t = A(\alpha\beta)^{-t} + \tilde{z}$ where the constant A can be identified using the terminal condition $z_T = 1$: $z_T = 1 = A(\alpha\beta)^{-T} + \tilde{z}$ or $A = (\alpha\beta)^T(1 - \tilde{z})$. Hence, z_t evolves as a monotonically decreasing function of time:

$$z_t = \frac{1 - (\alpha\beta)^{T+1-t}}{1 - \alpha\beta}.$$

   If the time horizon of agents is infinite ($T \to \infty$), $z_t = (1 - \alpha\beta)^{-1} = \tilde{z}, \forall t$, which implies (34). Consequently, d_t and c_t always grow at the same rate as y_t.

2. From (27), (28) and (34),

$$\frac{k_{t+1}}{y_t} = \alpha\beta\varphi(q_t). \tag{59}$$

Using the identity $k_{t+1}/y_t = (k_{t+1}/k_t)(k_t/y_t)$ and (29), (59) can be recast as (35). Using (28) again, $k_{t+1}/k_t = (\varphi(q_t)d_t)/(\varphi(q_{t-1})d_{t-1})$ and (35) then gives the expression growth factor of output (36). Given point 2) of Lemma 1, this is also the growth factor of d and c.

Appendix E: Proof of Lemma 2

Point 1 follows from (13) and (14): If the growth rate of q is positive , $q_t \to +\infty$ and $\chi(q)$, $\mu(q)$ and $\varphi(q)$ tend toward their respective asymptotic value. Point 2 follows obviously from point 1 and (30). Using point 1, (35) becomes (38) and the growth factor of output (36) has the same BGP value. From Lemma 1 and point 2 of Lemma 2, this is also the growth factor of c_t, d_t and x_t as stated in point 3.

To prove point 4, let us assume that material variables grow at a strictly positive rate: d would become infinitely large, which would violate the law of mass conservation (31). Hence, either d is constant along a BGP or its growth rate is strictly negative. If d is constant, Lemma 2 implies that all material variables are constant as well. From (31), the resource stock R_t is constant too; so is the extraction rate. If d grows at a negative rate, it tends towards zero and, from lemma 2, so do k, y, c and x. (31) then implies that $R_t \to M$ and the extraction rate tends towards 0.

Appendix F: Proof of propositions 1 to 4

1. Point 1 of Proposition 1 follows from point 4 of Lemma 2. To obtain point 2, we write -given Lemmas 1 and 2- the PBGP expression of (33) as follows:

$$H(\hat{q}_t) = \left(\frac{k_t}{y_t}\right)_{BGP} \left(\frac{c_t}{y_t}\right)_{BGP} \Psi.$$

   Since $k_t = \varphi d_t$, (34) and point 1 of the proposition imply that $H(\hat{q}_t) = \bar{\varphi}\alpha\beta(1-\alpha\beta)\Psi$, which leads to (39). Since function H is nil when $\hat{q} = 1$ and strictly increasing in $\hat{q}$, (39) defines a unique value of $\hat{q} > 1$. Point 3 follows straightforwardly from (35) with a unitary growth factor of capital. To obtain point 4, we use successively $d_t/y_t = \alpha\beta$, (30) and lemma 2 in order to rewrite (31) as follows

$$\mathcal{M} = R_t + \underline{\mu} d_t = R_t + \underline{\mu}\alpha\beta\frac{x_t}{\underline{\chi} + \underline{\mu}} = R_t\left[1 + \alpha\beta\frac{\underline{\mu}E_t}{\underline{\chi} + \underline{\mu}}\right].$$

   Along a PBGP, $E_t = \tilde{E}$ and the last equality above leads straightforwardly to the constant value of R_t given in (41). $\Sigma(\tilde{E})$ is monotonically decreasing in $\tilde{E}$, with $\Sigma(0) = 1$ and $1 > \Sigma(1) > 0$.

Using $\tilde{E} = x/R$ (resp. (30)) and (41) leads to (42) (resp. (43)).

2. Proposition 2 follows straightforwardly from the BGP value of y given by (43): in cases 1 and 2, this value tends towards $+\infty$, which reflects that material output then grows at a strictly positive rate along a BGP. In both cases, it is easy to show that the growth rate of knowledge remains given by (39). The growth factor of capital (or output) is

$$\left(\frac{k_{t+1}}{k_t}\right)_{BGP} = \frac{\alpha\beta\bar{\varphi}}{(\underline{\chi}+\underline{\mu})\Lambda(0)+h(\hat{q})} \quad \text{in case 1,}$$
$$\left(\frac{k_{t+1}}{k_t}\right)_{BGP} = \frac{\alpha\beta\bar{\varphi}}{h(\hat{q})} \quad \text{in case 2.}$$

3. Proposition 3: Positivity constraints in (45) and (46) are obvious (see definition of a PBGP). The upper bound on $\tilde{E}$ in (45) follows from (44) where d/y and $\hat{q}=1$ have been set to 1. The highest $\tilde{E}$ would indeed be reached if output was fully allocated to investment (unitary saving rate) and if the capital stock was allocated only to production activities (no research). Since the saving rate is in $[0,1]$, the left hand side of (44) lies in $[0,\bar{\varphi}]$ so that

$$0 < (\underline{\chi}+\underline{\mu})\Lambda(\tilde{E}) + h(\hat{q}) < \overline{\varphi}.$$

The left inequality will be necessarily satisfied since $\Lambda(\tilde{E}), h(\hat{q}) > 0$. The right inequality is equivalent to the right inequality in (46). Note then that if (47) does not hold, the upper bounds on $\tilde{E}$ and $h(\hat{q})$ in (45) and (46) are negative, i.e. no PBGP can exist.

4. Proposition 4: The condition on $h(\hat{q})$ follows from the constraint $\tilde{E} \geq 0$: from (40), this requires that $(\underline{\chi}+\underline{\mu})\Lambda(0) > \bar{\varphi}\alpha\beta - h(\hat{q}) > 0$, which is equivalent to (48). If (49) did not hold, the right-hand-side of (48) would be negative, $h(\hat{q})$ (and $\tilde{E}$) being constrained to be negative too.

Appendix G: Calibration

We assume the following functional forms which satisfy the model assumptions: $h(z_t) = [z_t - 1]^\gamma$ with $\gamma > 1$, $\psi(q_t) = q_t^\eta$ with $\eta > 0$ and

$$\mu(q_t) = \underline{\mu} + q_0\frac{\mu_0 - \underline{\mu}}{q_t}, \text{ with } \mu_0 > \underline{\mu} > 0$$
$$\chi(q_t) = \underline{\chi} + q_0\frac{\chi_0 - \underline{\chi}}{q_t}, \text{ with } \chi_0 > \underline{\chi} > 0$$
$$\varphi(q_t) = \overline{\varphi} - q_0\frac{\varphi_0 - \overline{\varphi}}{q_t}, \text{ with } 0 < \varphi_0 < \overline{\varphi}.$$

The saving rate is set to 20% and is used to determine α conditionally on the discount factor value: $\alpha = 0.2/\beta$. Given our assumption of a unitary depreciation rate, we consider that the length of a time period is 10 years. The model parameters are set to obtain a real interest rate of 22% over a 10-year period (2% on an annual basis) and to satisfy the existence condition of a PBGP.

β	α	γ	η	$\underline{\mu}$	$\underline{\chi}$	$\bar{\varphi}$	μ_0	χ_0	φ_0
0.83	$0.2/\beta$	2	0.9	0.015	0.055	0.386	$5\underline{\mu}$	$5\underline{\chi}$	$\bar{\varphi}/5$