Nesting horizontal and vertical differentiation: a note

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Abstract

We explore a variant of the Hotelling model which allows to nest horizontal and vertical differentiation into a unified setup whose key parameter is the relative natural market size of the firms. In this setup, equilibrium prices increase whenever population's disparity decreases. We also explore the properties of the model in the case of entry by a vertically differentiated product into an otherwise horizontally differentiated industry.

1 Introduction

In duopoly models of product differentiation, it is now customary to distinguish between vertical and horizontal differentiation according to whether consumers are unanimous or not in their ranking of the variants of a product when these variants are sold at the same price. Vertical differentiation prevails if at equal price all consumers opt for the same variant; horizontal differentiation prevails otherwise (for more details, see Beath and Katsoulacos, 1991). Under horizontal differentiation, when tastes are heterogenous, it is therefore possible to segment the population of consumers according to their preferred variant. We may then associate to each variant the group of consumers who prefer that variant over the other and define this group as its *natural market*. In the extreme case of vertical product differentiation, the natural market of one firm consists of the whole market while the other has a zero market share at equal price. By contrast, horizontal differentiation accomodates a very large class of natural markets' configurations. For instance, in the case of spatial competition *à la Hotelling* (Hotelling ,1929)*,* when firm 1 is located at the left extremity of the linear market while firm 2 stands at the other extremity of it, the market does not view one firm as more "desirable" on average than the other since both natural markets are exactly of equal size. At the other extreme, firms can be

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located in the linear market in such a manner that almost all consumers would prefer to buy from one of the two firms, in spite of the fact that these firms set the same price. This would be the case, again in the classical Hotelling location model, when firm 2 is located at the right extremity of the linear market while firm 1 now stands very close to it. Due to transportation costs, almost all consumers buy from firm 1 when it quotes the same price as firm 2. Thus, this situation corresponds very closely to the definition of vertical differentiation, even if, *sensu stricto*, it should fall into the alternative category. In all such hybrid cases, and although differently from vertical differentiation no variant holds a definite advantage over the other when horizontally differentiated, one may argue that the firm counting a larger number of consumers in its natural market should somehow benefit from a larger market power: on average, the market views this variant as more desirable. A first natural question therefore comes to mind: to which extent do differences in natural market sizes translate into different equilibrium market valuations for the product? Another natural question is whether, when the size of the natural market of a particular firm tends to the size of the whole market, the corresponding equilibrium prices tend monotonically to the prices prevailing at equilibrium of the corresponding vertical product differentiation market. Such a conclusion would then allow to nest vertical and horizontal product differentiation models in a natural way. We develop hereafter a duopoly model that allows to address these questions in a precise way.

To this end, we adapt the canonical Hotelling model to allow for natural markets of different sizes. In the symmetric linear model with firms located at the extremities of the unit interval, natural markets are defined by the $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ - intervals respectively. In order to allow for natural markets with different sizes, we then assume that the density differs from one interval to the other. Notice that in this model, a vertical configuration appears as a limiting case where the density of one of the intervals tends to zero while the density of the other tends to1. In this setup, we show that equilibrium prices display two key properties: first, the level of prices at equilibrium decreases with the disparities in natural market size. Second the equilibrium price differential increases with the disparities. In other words, the more unequal in size the natural markets, the fiercer price competition.

We also consider the problem of entry of a vertically differentiated product in a stylized way. More precisely we consider the scope for entry by a product with higher and lower absolute quality. By focusing on the extreme equilibrium configurations where the entrant either is blockaded or preempts the market, we show that the distribution of asymmetries among the two groups of consumers matters only for the entry exclusion case.

The model is introduced in section 2 while the equilibrium analysis in the case of two firms is provided in section 3. In section 4, we define an analog of vertical differentiation for the case of three firms and examine the entry problem at the light of this definition.

2   The model

Let $[0, 1]$ be the set of types of consumers and consider two firms, the first one being located at point 0 and the other at point 1. The density over the types in $T_1 = \left[0, \frac{1}{2}\right]$ is equal to μ and to $1 - \mu$ over the types in $T_2 = \left[\frac{1}{2}, 1\right]$. Figure 1 illustrates this situation. The preferences of consumers of type $x, x \in [0, 1]$, are defined by

$$U(x) = S - tx - p_1$$

when the consumers buy from firm 1(firm located at point 0), and by

$$U(x) = S - t(1 - x) - p_2$$

when the consumers buy from firm 2(firm located at point 1), with t denoting the unit transportation cost and p_i the price set by firm $i, i = 1, 2$. Notice that, at equal prices, all consumers in T_1 prefer to buy from firm 1 while all consumers in T_2 prefer to buy from firm 2 so that, when $\mu < \frac{1}{2}$, there is a majority of consumers who prefer buying from firm 2 than from firm 1 at equal prices, and vice-versa when $\mu > \frac{1}{2}$. In particular, when $\mu = 0$, *all* consumers prefer buying from 2, which corresponds to the (extreme) case of vertical differentiation and, when $\mu = \frac{1}{2}$, we obtain the (opposite extreme) case of symmetric horizontal differentiation. For different values of μ, we get hybrid cases of horizontal product differentiation, with a majority preferring to buy from firm 2 (resp. firm 1) than firm 1 (resp. firm 2) according as $\mu < \frac{1}{2}$ (resp. $\mu > \frac{1}{2}$).

In the following, we normalize the transportation cost t by putting $t = 1$. We shall also assume without loss of generality that $\mu < \frac{1}{2}$, so that there is a majority of consumers who prefer to buy from firm 2 than from firm 1 at equal prices. Finally, we assume that the constant S is large enough to guarantee that the market is covered.

# 3   Equilibrium analysis

Let $x(p_1, p_2)$ be the solution to

$$S - x - p_1 = S - (1 - x) - p_2,$$

namely,

$$x(p_1, p_2) = \frac{1}{2}(p_2 - p_1 + 1).$$

Notice that, if $p_1 < p_2$, we have $x(p_1, p_2) > \frac{1}{2}$, so that the interval $T_1 = \left[0, \frac{1}{2}\right]$ is included in the set of consumers who buy from firm 1 at prices (p_1, p_2). To this set, one must add the interval of types $\left[\frac{1}{2}, x(p_1, p_2)\right]$ corresponding to consumers who prefer to buy from firm 2 than from firm 1 at equal price, but who prefer

$1 - \mu$

μ

0   1/2   1
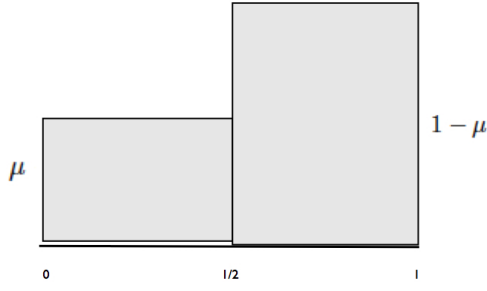
Distribution of tastes in the population of consumers

to buy from firm 1 at prices (p_1, p_2). Consequently, the demand $D_1(p_1, p_2)$ to firm 1 at prices (p_1, p_2) with $p_1 < p_2$ obtains as

$$D_1(p_1, p_2) = \frac{\mu}{2} + \frac{(1 - \mu)}{2}(p_2 - p_1).$$

Now, if $p_1 > p_2$, the point $x(p_1, p_2)$ is located at the left of $\frac{1}{2}$ and

$$D_1(p_1, p_2) = \frac{\mu}{2}(p_2 - p_1 + 1).$$

The corresponding revenue functions $R_i, i = 1, 2$, obtain as

$$R_1^l((p_1, p_2)) = (\frac{\mu}{2} + \frac{(1 - \mu)}{2}(p_2 - p_1))p_1$$

when $p_1 < p_2$, and

$$R_1^h((p_1, p_2)) = \frac{\mu}{2}(p_2 - p_1 + 1)p_1$$

when $p_1 > p_2$.

Notice that, since $\mu < \frac{1}{2}$, the demand function of firm 1 is linear convex with a kink at $p_1 = p_2$. Therefore, the revenue function might not be concave in own price.

The demand $D_2(p_1, p_2)$ and revenue functions $R_2(p_1, p_2)$ for firm 2 are easily derived as

$$D_2(p_1, p_2) = \frac{(1 - \mu)}{2} + \frac{\mu}{2}(p_1 - p_2)$$

$$R_2(p_1, p_2) = (\frac{(1 - \mu)}{2} + \frac{\mu}{2}(p_1 - p_2))p_2$$

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when $p_1 > p_2$, and

$$
\begin{aligned}
D_2(p_1, p_2) &= \frac{1-\mu}{2}(p_1 - p_2 + 1) \\
R_2((p_1, p_2)) &= (\frac{1-\mu}{2}(p_1 - p_2 + 1))p_2
\end{aligned}
$$

when $p_1 < p_2$. This revenue function is concave in own price.

Direct computations allow us to derive the firms' best replies from the first order conditions on the above revenue functions.

$$
\begin{aligned}
\phi_1^l(p_2) &= \frac{\mu}{2(1-\mu)} + \frac{p_2}{2} \text{ iff } p_1 \le p_2 \\
\phi_1^h(p_2) &= \frac{1}{2} + \frac{p_2}{2} \text{ iff } p_1 \ge p_2
\end{aligned}
$$

where the upperscript l (resp. h) denotes the configuration where p_1 is lower (resp. higher) than p_2 and

$$
\begin{aligned}
\phi_2^h(p_1) &= \frac{1-\mu}{2\mu} + \frac{p_1}{2} \text{ iff } p_1 \ge p_2 \\
\phi_2^l(p_1) &= \frac{1}{2} + \frac{p_1}{2} \text{ iff } p_1 \le p_2.
\end{aligned}
$$

Notice that since $\mu < \frac{1}{2}$, we have that $\frac{\mu}{1-\mu} < 1$. It follows that $\phi_1^l < \phi_1^h$ whereas $\phi_2^l > \phi_2^h$. The shape of firms' best replies in the case where $\mu < \frac{1}{2}$ are summarized in Figure 2.

Notice that firm 2's best reply is continuous but kinked. By contrast, since $\frac{\mu}{1-\mu} < 1$, there are possibly two candidate best replies for firm 1 against p_2, one in the region where $p_1 > p_2$ and one where $p_1 < p_2$. It is a matter of computation to establish that there exists a discontinuity in firm 1's best reply at some unique price $\tilde{p}_2$. Firm 1's best reply jumps down at this price. Formally, we solve

$$
R_1^l((\phi_1^l(p_2), p_2)) = R_1^h((\phi_1^h(p_2), p_2))
$$

for p_2 in order to identify the critical price level of firm 2 for which firm 1 is indiffernt between setting a high price and sell only on its natural market or undercut firm 2 and capture part of its (bigger) market. We obtain

$$
\tilde{p}_2 = \frac{\sqrt{\mu}}{\sqrt{1-\mu}}
$$

Since firm 1's best reply correspondence exhibits a downward jump, the existence of a pure strategy equilibrium is not guaranteed.

We first identify the unique candidate pure strategy equilibrium. Given the firms' best replies, it is first clear that there exists no symmetric pure strategy equilibrium. Additional computations directly show that there exists no asymmetric equilibrium in which firm 2 quotes the lowest price. The only remaining candidate is such that $p_1 < p_2$. Combining ϕ_1^l and ϕ_2^l one obtains
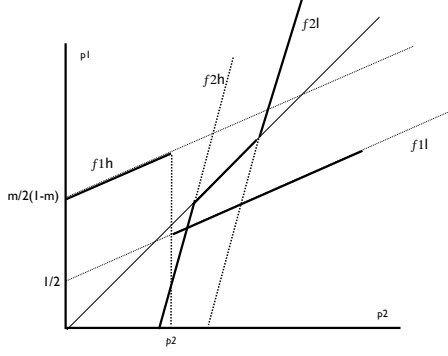
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Figure 1: example caption

$$p_1^* = \frac{\mu + 1}{3(1 - \mu)}, p_2^* = \frac{2 - \mu}{3(1 - \mu)}.$$

It is then easy to check that $p_1^* = \frac{\mu+1}{3(1-\mu)} < p_2^* = \frac{2-\mu}{3(1-\mu)}$ holds if, and only if, $\mu < \frac{1}{2}$. In other words, the equilibrium candidate indeed yields the desired price hierarchy. In addition, we check that $\tilde{p}_2 = \frac{\sqrt{\mu}}{\sqrt{1-\mu}} < p_2^* = \frac{2-\mu}{3(1-\mu)}$ whenever $\mu < \frac{1}{2}$. Accordingly, the equilibrium price for p_2 indeed belongs to the relevant segment of the best reply and the equilibrium exists. Summing up we have established the following proposition:

Proposition 1 *Suppose* $0 < \mu < \frac{1}{2}$, *there exists a unique Nash equilibrium given by*

$$p_1^* = \frac{\mu + 1}{3(1 - \mu)}, p_2^* = \frac{2 - \mu}{3(1 - \mu)}.$$

We may consider now the equilibrium prices corresponding to the "extreme" situations in which either *all* agents prefer to buy from firm 2 than from firm 1 at equal prices ($\mu = 0$: vertical product differentiation), or half of them prefer to buy from firm 1 and half of them from firm 2 under the same condition ($\mu = \frac{1}{2}$: symmetric horizontal product differentiation). In the first case, we get

$$
\begin{aligned}
D_1(p_1, p_2) &= \frac{1}{2}(p_2 - p_1)) \\
D_2(p_1, p_2) &= \frac{1}{2}(p_1 - p_2 + 1),
\end{aligned}
$$

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with corresponding profits

$$\pi_1(p_1, p_2) = \frac{1}{2}(p_2 - p_1)p_1$$
$$\pi_2(p_1, p_2) = \frac{1}{2}(p_1 - p_2 + 1)p_2.$$

The corresponding price equilibrium is easily derived from the first order conditions, namely,

$$p_1^* = \frac{1}{3}, p_2^* = \frac{2}{3}.$$

In the second case, we notice that, when $\mu = \frac{1}{2}$, equilibrium prices are equal to each other and equal to one.

We also notice that $\lim_{\mu \to 0} \{p_1^*(\mu)\} = \frac{1}{3}$ and $\lim_{\mu \to 0} \{p_2^*(\mu)\} = \frac{2}{3}$: when μ tends to zero, the model gets closer and closer to a situation of vertical differentiation, in which a larger and larger majority prefers variant 2 to variant 1 ($1 - \mu$ tends to 1), and the corresponding equilibrium prices converge to the equilibrium prices in the limit model.

It is easy to check that the equilibrium analysis covering the case when $\frac{1}{2} > \mu > 1$ is, *mutatis mutandis*, identical to the preceding one: firm 1 now plays the role of firm 2 in the definition of demands and profits, firm 2 selling now the variant preferred by the majority. Additional properties of the equilibrium are worth being mentioned.

Proposition 2 *The equilibrium price differential decreases with μ, while absolute price levels both increase with μ.*

In other words, a larger symmetry in the population's tastes relaxes price competition. The intuition is straightforward: a larger μ means that the natural market of firm 2 gets bigger which implies that it is less attractive to challenge the other's natural market in relative terms. Market valuations of the product tend to reflect the distribution of tastes among variants in the population. Equilibrium market valuations reflect the disparities in natural market sizes.

4 Entry

An interesting extenssion of the approach developed above consist in extending our model to a triopoly. It turns out however that the extension is far from immediate. In the present note, we focus instead on the more simple case of entry. More precisely, we shall consider an analog for the concept of vertical differentiation when there are more than two firms and that there is no unanimity of the population concerning the ranking of the variants sold by these two incumbent firms. Usually, the treatment of vertical differentiation with more than two variants assumes that *all* variants can be unanimously ranked by the population (see, for instance, Gabszewicz and Thisse (1980) or Shaked and Sutton

(1986)). This question is particularly meaningful when entry occurs in a market with two existing firms *not* unanimously ranked, but with natural markets of different size, as in the preceding section. We concentrate on entry by inferior quality and entry by superior quality. These cases constitute a natural first step to extend the problem of *natural oligopolies,* which appears when variants can be unanimously ranked by the agents, to new situations where the variants sold by the incumbent firms are horizontally differentiated.
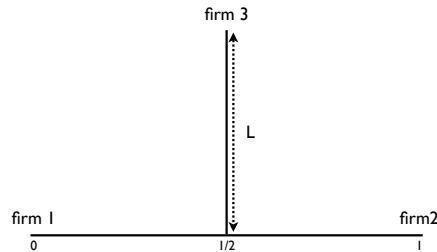
firm 3

L

firm I   firm2
0   1/2   1

Figure 2: The Case of Entry

4.1 Entry by inferior quality

To this end, assume first that a third firm, - firm 3 - , is contemplating to enter the market at the end of a linear segment of length L (see figure 3). We assume that there are no consumers located along this segment. Accordingly, the entrant's natural market is always equal to zero. In order to capture the idea that firm 3 enters with an inferior quality, we have to formalize a "quality" such that no consumer in the existing market served by the incumbents (firms 1 and 2) would prefer to buy from firm 3 rather than from firms 1 or 2, when the three prices p_1, p_2 and p_3 are equal. Formally, this amounts to that $L > \frac{1}{2}$. This assumption implies indeed that even the consumer who is located the closer to firm 3 localized at the extremity of its market L, is not willing to buy from it when the three prices p_1, p_2 and p_3 are equal. Thus, and *a fortiori*, at equal prices, *all* consumers located along the line $[0, 1]$ incur more utility when moving to either firm 1 or firm 2 rather than moving to firm 3. Since all prices are equal, each of them chooses to buy either from firm 1 or firm 2, but never from firm

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3. We now identify a sufficient condition on the value of L guaranteeing that firm 3 cannot enter the market at the "pre-entry" equilibrium prices (p_1^*, p_2^*) of firms 1 and 2 identified above.

Let firm 3 quote a price p_3 equal to 0 and firms 1 and 2 choose their "pre-entry" equilibrium prices (p_1^*, p_2^*). In this case, the utility obtained by any consumer x buying at firm 1 is equal to $(S - x - p_1^*)$ and his utility from buying at firm 3 is equal to $S - (L - x)$. Consequently, whenever the inequality

$$S - x - p_1^* \geqslant S - (L - x)$$

holds for all consumers x, no consumer buying from firm 1 at a price equal to p_1^* would be willing to buy from 3 at price p_3 equal to 0. Substituting the value for p_1^* in the above inequality, we get

$$S - x - \frac{\mu + 1}{3(1 - \mu)} \geqslant S - (L + x - \frac{1}{2})$$

for all x. Considering the value of x which is the closer from L, namely $x = \frac{1}{2}$, this inequality rewrites as

$$L \geqslant \frac{\mu + 1}{3(1 - \mu)} + \frac{1}{2},$$

which constitutes a sufficient condition for barring entry to firm 3. Indeed, this inequality guarantees that the consumer who is the closer to firm 3 located at L is not willing to buy from it. This is *a fortiori* true for consumers buying from firm 2 at price p_2^* since p_2^* is even larger than p_1^*. Furthermore, since by their very definition, the prices p_1^* and p_2^* are mutual best replies for firms 1 and 2, the vector of prices $(p_1^*, p_2^*, 0)$ is a price equilibrium at which the entry of firm 3 is blockaded. Thus

Proposition 3 *At any value of L in the domain $\left[\frac{\mu+1}{3(1-\mu)} + \frac{1}{2}, \infty\right)$ firm 3 is excluded from the market at the vector of prices $(p_1^*, p_2^*, 0)$, which is a price equilibrium.*

Notice that the bound $\frac{\mu+1}{3(1-\mu)} + \frac{1}{2}$ is increasing with μ : the larger μ, the larger the bound. When μ is equal to $\frac{1}{2}$ (horizontal differentiation), the corresponding value satisfying the bound constraint is $L > \frac{3}{2}$. Similarly, when μ is equal to 0 (vertical product differentiation), the inequality reduces to $L > \frac{5}{6}$. In the hybrid cases, the constraint becomes more and more stringent in proportion as μ increases from 0 to $\frac{1}{2}$. In other words, less disparity among incumbents makes the entry of an inferior quality product less likely.

4.2 Entry with a product of higher quality

Let us now consider the scope for entry by a third firm selling a product of intrinsic quality, say $\hat{S}$. This firm is located at the extreme of the third arc and

we normalize its length to $\frac{1}{2}$. We assume that consumers' preferences towards this variant are given by

$$U^x(\hat{S}, p) = \hat{S} - (1 - x) - p_3 \ , \ \forall x \in [0, 1]$$

$$\hat{S} > S$$

which amounts to assume that the intrinsic quality attached to variant 3 is larger than the intrinsic quality of variants 1 and 2. Notice however that one can properly speak of vertical differentiation only if the $\hat{S} - S > 1$, i.e. the intrinsic quality difference more than covers the transportation cost differential for *all* consumers.

Now we look for the conditions under which the triplet : $(0, 0, p_3^*)$, with p_3^* the highest value of p_3 such that $D_1(p_3, 0, 0) = D_2(p_3, 0, 0) = 0$ is a triopoly equilibrium: at this equilibrium, firms 1 and 2 are excluded from the market. Even quoting a zero price they are unable to have a strictly positive market share (limit pricing strategy). By definition, we have

$$p_3^* = \hat{S} - (S + 1)$$

In order to identify the conditions under which this market preemption equilibrium prevails, we simply have to compare firm 3's payoffs in the triopoly equilibrium, with all the three firms active in the market. Notice that in an equilibirum with three active firms, it must be the case that $p_1 = p_2$. The distribution of tastes in the population, as summarized by the parameter μ does not matter anymore since, from the point of view of firm 3, capturing consumers belonging to the natural market of 1 or those belonging to the natural market of 2 is exactly symmetrical. The differences in market densities accordingly cancel out in the equilibrium payoffs.

Straightforward computations allow us characterize the triopoly candidate equilibrium as follows:[1]

$$p_1^* = p_2^* = \frac{S + 2 - \hat{S}}{6}$$

$$p_3^* = \frac{1 - S + \hat{S}}{6}$$

By comparing the resulting profit of firm 3 in this equilibrium to those accruing from the limit pricing strategy $p_3^* = \hat{S} - (S + 1)$, it is immediate to show that there exists a lower bound for $\hat{S}$, strictly larger than $S + 1$ beyond which market preemption is the unique equilibrium. The key property here is that the lower bound does not depend at all on μ. Consequently, entry by low or high quality

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[1]Notice that the existence of this equilibrium candidate is not guaranteed. When the disparity in group sizes is very large, the firm that sells on the smaller natural market is likely to deviate by undercutting in order to invade the other's natural market. Such a deviation is likely to destroy the pure strategy equilibrium candidate. For the sake of simplicy, we shall assume that an equilibrium always exists.

does not operate in the same manner since the preentry equilibrium depends on the value of μ when entry takes place with a lower quality variant, while it does not when entry occurs with a variant of higher quality than those sold by the incumbent firms. This has to be contrasted with entry in a market with all firms being completely and unanimously ranked, like in Gabszewicz and Thisse,1980 or Shaked and Sutton,1986. In this case, the condition for exit with a variant at the bottom of the quality ladder due to the entry of a new variant at its top is equivalent to the condition which prevents entry of the bottom quality when the top variant is already in the market.

5   Final Remarks

In this note, we have explored a variant of the Hotelling duopoly model where firms' natural markets differ in size. This allows us to study firms' relative market power as a function of the relative size of their natural market. In this context, vertical differentiation obtains as a limiting case where one firm's natural market shrinks to zero. In this setup, we show that in equilibrium, prices are increasing in the degree of symmetry so that a vertically differentiated market is always more competitive than a horizontally differentiated one. In order to generalize the approach to more than two firms, we have considered the scope for entry by a vertically differentiated firm. We leave for future research the analysis of oligopolistic competition that would result from considering a collection of n firms, each being defined by its own natural market.

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