Cooperation, competition and market entry*

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Abstract

We propose a model of network formation among competitors in a Tullock contest. Agents first form their partnerships and then choose their investment in the contest. By cooperating with a competitor, an agent increases his valuation for the prize, but he also increases the valuation of his rival. It is thus not obvious that competitors decide to cooperate. We find that the network formation process can act as a barrier to entry. The pairwise equilibrium network features a group of completely interconnected agents and another group of isolated agents who choose not to participate to the contest. Barriers to entry may either hurt total surplus as the winner of the prize does not exploit all the possible network benefits, or improves it since the wasted efforts are smaller when competition is less fierce. When networking acts as a barrier to entry, pairwise equilibrium networks are inefficient.

1 Introduction

In many instances in economics and politics competition takes form in contests where competing agents spend resources in order to increase their chance of winning a prize. For instance, firms invest in R&D in order to get a patent and invest in marketing campaign to increase their market shares, colleagues work hard when they are competing for a promotion, lobbies exert pressure to influence political

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decisions, etc.[1] Rivals in contest are often involved in bilateral cooperative relationships. Arzaghi and Henderson (2008) report that small advertising agencies share innovative ideas and expertise while competing to obtain new accounts. Competing firms are increasingly sharing databases concerning detailed customer information (Liu and Serfers, 2006; Leminen et al., 2008). Universities cooperate in developing joint IT projects, opening their libraries to each others' students or sharing software licence.

This paper develops a model of network formation among competitors in a Tullock contest. Agents first form their partnerships and then choose their investment in the contest. By cooperating, two competitors increase their valuation for the prize, say because they have access to a larger market in case they get the prize, or because they share information on how to exploit the prize. We assume that the linking costs are negligible. By forming a link with a competitor, an agent improves the position of his rival. It is thus not obvious that competitors wish to form links, even if there is no cost attached to doing it. We have chosen to focus on small costs of link formation to analyze the case that favors most collaboration, and show that even in that case, market mechanisms deters collaboration.

For each network structure defining the profile of valuations of the competitors, there is a unique Nash equilibrium choice of effort in the second stage. Solving the game by backward induction, we then characterize the set of pairwise equilibrium networks of the link formation game. We show that any two unconnected agents that are participating to the contest are better off by adding a link (Proposition 1). We establish that the only pairwise equilibrium networks are group dominant networks, where one group of agents is completely linked to the other agents in the group, and the remaining agents have no links and decide not to participate to the contest (Proposition 2). Network formation can thus act as a barrier to entry to the contest. We show that whenever networking acts as a barrier to entry there is a conflict between stability and efficiency, as no pairwise equilibrium network maximizes total surplus (Proposition 3). Finally, we show that the welfare effect of barriers to entry is not clear. In some cases, endogenous barriers to entry hurt total surplus because the winner of the prize does not exploit all the possible network benefits. In others it helps improve the welfare because the total wasted efforts are smaller when competition is less fierce.

[1] See Konrad (2009) for an excellent overview on contest theory and its applications.

From an empirical point of view, the observed network structures of collaboration among firms are never complete. Often times they display asymmetries, clusters of strongly linked rivals and agents who decide not to cooperate at all (see for instance Hagedoorn and Schakenraad, 1992 and Powel et al., 2005). Hochberg et al. (2010) show that strong network ties among venture capitalists in a given market deters entry. Our theoretical predictions support these observations.

This paper contributes to the literature on competition in networks. That literature has focused on horizontal cooperation between oligopolists or between rival firms in R&D races. Goyal and Moraga (2001) propose a game where competing firms in the final market form links to cooperate in R&D leading to smaller cost of production. They show by means of an example that asymmetric networks may lead to exclusion and if so, it does not necessarily harm total surplus. Goyal and Joshi (2003) develop a model where each partnership translates exogenously and linearly into a marginal cost reduction. Assuming that the parameters are such that all firms are active in the market, they show that the only stable network is the complete network when the costs of link formation are small.[2] Goyal and Joshi (2006) introduced a model of patent race in networks, where the period at which a firm expect to innovate is positively correlated to its number of links. They show that the only stable network architecture is the complete network when the cost of links formation is small. Marinucci and Vergote (2011) show that group dominant networks where unconnected agents are left out of the competition are the only pairwise stable networks in an all pay auction in which link formation affects the value of the prize in a multiplicative way.

Our results about the stable network architecture when partnerships are bilateral thus confirm previous ones in a different framework. There are four main distinctions between our approach and others in the field. First, the competition occurs through a contest in our model, while in the other papers it occurs through standard quantity or price competition in the final market, or through an auction. Second, the nature of partnerships differs. In our model, two partners either have a higher valuation for the prize, or are more efficient in providing efforts in the contest when they collaborate. In other papers, partners reduce their marginal cost of production either directly, indirectly through R&D collaboration, or through cross licensing. Third, the effort of the agent is endogenous in our model and affected by the network. Fourth, we

[2] Notice that we would reach the same conclusion if we assumed participation by Proposition 1.

do not assume that the parameters are such that each agent participate to the competition but instead analyze in detail the question of entry and participation.

Marinucci and Vergote (2011) were the first to explain the formation of asymmetric groups without assuming prohibitive cost of link formation leading to market exit. Market exit is not only the consequence but also the cause of the formation of asymmetric networks. Indeed, insiders always want to form as many links as possible. On the other hand outsiders that are facing a strong group of completely connected competitors do not want to add a new link, even if this link is not costly. We show in this paper that their result are also valid in the Tullock contest, where the value of link formation is linear rather than multiplicative. Contrary to Marinucci and Vergote (2011), the total surplus is not necessarily reduced when some agents are left out of the market. More competition between stronger agents does not always lead to higher surplus in our model because as the network becomes more dense, the sum of wasted efforts may increase more than the expected valuation of the winner of the contest.

Westbrock (2010) studied efficiency in the model of Goyal and Joshi (2003). He found that strongly efficient networks must either be group dominant or have the interlinked star architecture. In our model, we show that group dominant networks never maximize total surplus, but the smallest group dominant networks may lead to higher surplus than others pairwise equilibrium network.

Other papers have also introduced contests in networks. Jost (2007) studies a Tullock patent contest where firms choose their R&D investment to win the competition. When firms form a link they share their R&D capacities, leading to free-riding and underinvestment in R&D. In equilibrium all players are active and have exactly one link. Hiller (2012) proposes a model of signed network formation, where agent extend positive links to others in order to extract rents from enemies. He finds that the network structure that emerges features asymmetric groups of agents, with members from bigger group extracting rents from those in the smaller. Franke and Ozturk (2009) propose a model where the network exogenously determines the conflictive relations among agents, and relate the network structure to the conflict intensity.

We equally contribute to the theoretical literature on Tullock contests. Stein (2002) has shown that when the valuation of a player increases, his expected payoff increases provided no player leaves the contest. We show that this remains valid when the valuation of two players increases by the same amount, even if some player is initially in a better position in the contest, and if some players decide to leave the contest. Eggert and Kolmar (2006) analyze a contest model where the total prize depends on the number of participants. In our model, the value of the prize depends on the number of links of the player that wins the contest.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we characterize pairwise equilibrium networks. Section 4 analyzes efficiency and contrast the efficient networks to the pairwise equilibrium networks. Section 6 concludes.

2 Model and notation

2.1 Networks

Let $N = \{1, 2, ..., n\}$ be the finite set of agents. Each agent announces the set of links he would like to form, and the links that forms are those where both agents have announced their intention to form that link. We let $\sigma_{i,j} = 1$ if player i intends to form a link with player j while $\sigma_{i,j} = 0$ if he does not. The strategy of player i is $\sigma_i = \{\{\sigma_{i,j}\}_{j \in N \setminus \{i\}}\} \in S_i$. When $\sigma_{i,j} = \sigma_{j,i} = 1$, a link between i and j is formed. We write $g_{i,j} = 1$ when a link between i and j exists and $g_{i,j} = 0$ otherwise. A network $g = \{(g_{ij})\}$ is the list of which pairs of individuals are linked to each other. Let g^N be the collection of all subsets of N with cardinality 2, so g^N is the complete network. The set of all possible networks on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. A strategy profile $\sigma = (\sigma_i)_{i \in N}$ therefore induces a network $g(\sigma) \in \mathbb{G}$. The network obtained by adding the link ij to an existing network g is denoted $g + ij$ and the network that results from deleting the link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \in N \mid \exists j \text{ such that } ij \in g\}$ be the set of agents who have at least one link in the network g. Let $N_i(g)$ be the set of agents that are linked to i : $N_i(g) = \{j \in N \mid ij \in g\}$. The degree of agent i in a network g is the number of links that involve that agent: $d_i(g) = \#N_i(g)$. A path in a network $g \in \mathbb{G}$ between i and j is a sequence of agents $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j. A component h of a network g is a nonempty subnetwork $h \subseteq g$

satisfying (i) for all $i \in N(h)$ and $j \in N(h)\backslash\{i\}$, there exists a path in h connecting i and j, and (ii) for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. Given a set of players $S \subsetneq N$, a network g^S is such that the agents in S are connected to each other agents in S and the agents in $N\backslash S$ have no link. A network with that structure is called a dominant group network. We sometimes say that members in S are completely connected when every link is formed among these agents.

2.2 Model

We consider a society of n ex-ante identical individuals who first form links of collaboration and then choose their level of effort in order to win a prize in a contest. The prize is allocated to the individuals according to the profile of efforts of all agents. We assume that agent i's probability of winning the prize p_i is given by the ratio of his effort (e_i) and the sum of all efforts $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$. The cost of providing effort is assumed to be equal to the level of effort. The valuation of agent i for the good is decomposed into a fixed component v and a variable component that depends on the degree of the agent $v_i(g) = v + d_i(g)\beta$. Both the fixed valuation v and the impact of a link on the valuation of a player β is common to each agent and does not depend on the number of links the agent has. Assuming that each link between agents increases linearly their value of winning the contest is equivalent in our framework to assuming that the marginal cost of effort in the contest decreases linearly with additional links.[3] Our setup thus fits both the situation where cooperation increases the value of the prize and the situation where it reduces the cost of contest effort. We restrict our analysis in this paper to the case where the cost of link formation c is positive but close to 0. Forming cross-licensing agreements, establishing joint R&D programs or sharing valuable information certainly involves costs in terms of organization, administration, layers, etc. However, we believe that in many applications we have discussed in the introduction the value of the prize is so important that these costs do not explain the partnership choices of the agents. There are other costs to link formation that are related to the competitive effect of collaboration since by forming a link with a competitor, an agent improves the

[3] Firms that are more connected may have more information on how to conduct a successful marketing campaign, they may share the costs of an investment, they may sign cross-licensing agreements, etc. See Corchon (2007) for the equivalence between the two formulations of the model.

position of its rival. It is thus not obvious that competitors wish to form a link, even if there is no cost attached to doing it. The purpose in this paper is to analyze in detail this question. We however assume that links are not free to avoid a situation where firms are involved in some partnership but do not plan to participate to the contest and as a result do not derive any benefit from these collaborations.

Given a network g, the expected payoff of agent i is given by

$$\Pi_i(e_i, e_{-i}, g) = p_i(e_i, e_{-i})(v + d_i(g)\beta) - e_i - cd_i(g), \tag{1}$$

2.3 Contest stage

Given a network g, an agent i chooses a nonnegative effort e_i in order to maximize his payoff $\Pi_i(e_i, e_{-i}, g)$. We do not exclude corner solutions to this problem. There are network architectures and parameters configurations such that the optimal effort of an agent is nil at the Nash equilibrium of the contest stage. Let us introduce the **ordering function** $\phi : N \times \mathbb{G} \to N$ to rank the players according to their valuation in the network so that the players with a higher valuation have a lower index in the reordering. When multiple players have the same valuation, the players with lower indices in the player set have a lower rank in the ranking. It follows that $\phi(k, g) = 1$ if $v_k(g) \geq v_l(g)$ for all $l \in N$ and $k < l$ for all l such that $v_k(g) = v_l(g)$.

From Hillman and Riley (1989), the number of participating agents is the largest $\phi(k, g)$ such that $v_k(g) > (\phi(k, g) - 1)/(\sum_{j:\phi(j,g)\leq\phi(k,g)} 1/v_j(g))$. We let $\kappa(g)$ be the value of $\phi(k, g)$ that solves this problem, and note the set of agents participating to the contest in the network g by $K(g)$. Thus, the $\kappa(g)$ players with the highest valuation at the network g participate in the contest, while the remaining agents do not. In particular, for k such that $\phi(k, g) \leq (>)\kappa(g)$, we have

$$v_k(g) > (\leq) \frac{(\kappa(g) - 1)}{\sum_{j:\phi(j,g)\leq\kappa(g)} 1/v_j(g)} = h_{\kappa(g)}(g)\frac{(\kappa(g) - 1)}{\kappa(g)},$$

where $h_{\kappa(g)}(g) = \kappa(g)/(\sum_{j:\phi(j,g)\leq\kappa(g)} 1/v_j(g))$ is the **harmonic mean** of the largest $\kappa(g)$ valuations.

Following Stein (2002), in a network g, the Nash equilibrium level of efforts is given by:

$$e_i^*(g) = \begin{cases} \frac{\kappa(g)-1}{\kappa(g)} h_{\kappa(g)}(g)(1 - \frac{\kappa(g)-1}{v_i(g)} \frac{h_{\kappa(g)}(g)}{\kappa(g)}) \text{ if } \phi(i,g) \leq \kappa(g) \\ 0 \text{ if } \phi(i,g) > \kappa(g) \end{cases}, \quad (2)$$

so that the probability that agent i wins the contest is

$$p_i^*(g) = \begin{cases} 1 - \frac{\kappa(g)-1}{v_i(g)} \frac{h_{\kappa(g)}(g)}{\kappa(g)} \text{ if } \phi(i,g) \leq \kappa(g) \\ 0 \text{ if } \phi(i,g) > \kappa(g) \end{cases}, \quad (3)$$

and the equilibrium payoff is

$$\Pi_i(e_i^*, e_{-i}^*, g) = \begin{cases} v_i(g) p_i(g)^2 - c d_i(g) \text{ if } \phi(i,g) \leq \kappa(g) \\ -c d_i(g) \text{ if } \phi(i,g) > \kappa(g) \end{cases}. \quad (4)$$

When the number of participants is *fixed*, Stein (2002) has shown that both the equilibrium probability of getting the prize and the equilibrium payoff of an agent is increasing in his valuation. In our framework, the agents have the ability to increase their valuation by forming links. However, when they create a new link, both their valuation and the valuation of a competitor increase so that the net effect on the payoff of the agent is unclear. In addition, by forming cooperative links, rivals may obtain an indirect benefit through the reduction of effort or the exclusion of less connected agents. We analyze the formation of networks of collaboration in the following section.

3 Pairwise equilibrium networks

For each network structure defining the profile of valuations of the competitors, there is a unique Nash equilibrium choice of effort in the second stage. Solving the game by backward induction, we characterize the set of pairwise equilibrium networks of the link formation game.

Let $\Pi_i(e_i^*(g(\sigma)), e_{-i}^*(g(\sigma)), g(\sigma))$ be the payoff of agent i in the network $g(\sigma)$ induced by the strategy profile $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ when the agents chose their optimal contest effort e^* in the second stage.

A strategy profile $\sigma^* = \{\sigma_1^*, \sigma_2^*, ..., \sigma_n^*\}$ is a Nash equilibrium of the link formation game if $\Pi_i(e_i^*(g(\sigma^*)), e_{-i}^*(g(\sigma^*)), g(\sigma^*)) \geq \Pi_i(e_i^*(g((\sigma_i, \sigma_{-i}^*))), e_{-i}^*(g((\sigma_i, \sigma_{-i}^*))), g((\sigma_i, \sigma_{-i}^*)))$ for all $\sigma_i \in S_i$, for all $i \in N$. As is standard in the theory of network formation since the seminal paper by Jackson and Wolinsky (1996), we add to the notion of Nash equilibrium the requirement that there does not exist a pair of agents that would like to form a link. Pairwise equilibrium are Nash equilibrium satisfying this additional requirement. Let $\pi_i(g) = \Pi_i(e_i^*(g(\sigma)), e_{-i}^*(g(\sigma)), g(\sigma))$ for some σ such that $g = g(\sigma)$.

Definition 1. A network g is a pairwise equilibrium network if

1. there is a Nash equilibrium σ^* that induces g.

2. for all $ij \notin g$, if $\pi_i(g) < \pi_i(g + ij)$ then $\pi_j(g) > \pi_j(g + ij)$.

We show in Proposition 1 that every network g such that two agents participate but are not connected is not a pairwise equilibrium.

Proposition 1. *Every network $g \subseteq g^N$ such that $i, j \in K(g)$ and $ij \notin g$ is not a pairwise equilibrium.*

All the proofs are in the appendix.

We decompose the proof of the proposition in 4 steps. In Step 1, we show that the set of participating agents in the network $g+ij$ is a subset of the set of participating agents in the network g. In Step 2, we show that the agents i and j participate in $g + ij$. In Step 3, we show that the players i and j are better off by adding the link ij if the set of participating agents is the same under the networks g and $g + ij$. Finally, in Step 4, we show that if the set of participating agents is smaller under the network $g + ij$ than under g, then either the players i and j are better off by adding the link ij, or some player k is connected to i in the network g but does not participate to the contest and is thus better off by deleting the link ik.

In a pairwise equilibrium, we thus have a group of players who are connected among themselves and participate to the contest and another group of agents who are isolated and are better off by not participating to the contest. Hence the pairwise equilibrium networks are dominant group networks g^S for some $S \subseteq N$. We characterize the set of pairwise equilibrium in Proposition 2.

Proposition 2. *A network g is a pairwise equilibrium if*

(i) $g = g^N$,

(ii) $g = g^S$ *for all* $S \subseteq N$ *such that* $3/2 + \sqrt{5/4 + v/\beta} \leq s \leq n-2$,

(iii) $g = g^{N \setminus \{i\}}$ *for all* $i \in N$ *when either* $\frac{(v+\beta)(n-2)}{v+(n-2)\beta} + \frac{(v+\beta)}{(v+(n-1)\beta)} \leq n-2$, *or* $n \geq (v/\beta)^{1/2} + 2$ *and* $(v + (n-2)\beta)(1/(n-1))^2 > (v + (n-1)\beta)(1 - \frac{(n-1)}{\frac{(n-2)(v+(n-1)\beta)}{v+(n-2)\beta} + 1 + \frac{(v+(n-1)\beta)}{v+\beta}})^2$.

The complete network is always a pairwise equilibrium. If an agent deviates from the complete network by cutting links, he either reaches a network g' where he is not participating or where he is participating but not connected to some participating agents. In both cases he is better off by maintaining his links. Smaller group dominant networks are also pairwise equilibria if the the relative weight of partnerships in determining the valuation of agents for the prize is sufficiently high (v/β is small) or if the population size is high. The higher the relative importance of collaboration in determining the valuation, the smaller the dominant group can be in a pairwise equilibrium. In a group dominant network, the higher the size of the group, the smaller the incentives of unconnected agents to participate because as the size of the group increases, there are more competitors and they each have a higher valuation for the prize. When the size of the group is larger than the threshold value ($3/2 + \sqrt{5/4 + v/\beta}$), no pair of unconnected players would participate by adding a link. If the group dominant network g^S is a pairwise equilibrium, each group dominant network g^T is a pairwise equilibrium as long as $\#T \geq \#S$. If not and the group S is composed of $n-2$ agents, then the condition to determine whether the network g^T that connects $n-1$ agents is a pairwise equilibrium changes. The network g^T is a pairwise equilibrium either if the isolated agent would not participate by adding a link with a participating agent, or if an agent from the group prefers not to add the link with the isolated agent.

4 Efficiency

A network is efficient if it maximizes the sum of payoffs of the agents when they choose their optimal effort in the second stage. In this section, we discuss the relationship between the network architecture and the total surplus. Notice that the total surplus is increasing in the expected valuation of the player getting the prize and decreasing in the total wasted efforts: $W(g) = \Sigma_{i\in N} p_i^*(g)v_i(g) - \Sigma_{i\in N} e_i^*(g)$. Inefficiencies may arise because the network benefits are not totally exploited, because the prize is allocated with some probability to agents that do not have the maximal valuation for the good, and because resources are wasted to influence the allocation of the prize. Using (3), the expected benefits of the agents in the contest is given by $\Sigma_{i\in N} p_i^*(g)v_i(g) = \Sigma_{i\in K(g)} v_i(g) - (\kappa(g)-1)h_{\kappa(g)}(g)$. It is increasing in the number of links of the participating agents and decreasing in the harmonic mean of the valuation of the participating agents. Using (2), the sum of efforts is given by $\Sigma_{i\in K(g)} e_i^*(g) = (\kappa(g)-1)h_{\kappa(g)}(g)/\kappa(g)$. It is increasing in the harmonic mean of the valuation of the participating agents.

It follows that for a fixed number of links and given a set of participating agents, the higher the harmonic mean of the valuation of the participating agents, i.e. the more equal the distribution of links among participating agents, the higher the sum of efforts and the lower the expected valuation of the agent getting the prize. For instance, if the set of participating agents is equal in two networks g and g' and the valuation profile of the participating agents in the network g' is a mean preserving spread of the valuation profile of the participating agents in the network g, then the sum of payoffs is higher under the network g' than under the network g.

The pairwise equilibria can be ranked in terms of total surplus (see Lemma 3 in the appendix). In smaller group dominant networks, the valuation of the agent getting the prize is smaller but the sum of wasted effort is also reduced since fewer agents compete for the prize and they each value less the good. When the common valuation of agents for the good is important relative to the network benefits ($v/\beta > 1$), smaller group dominant networks produce more surplus than bigger ones. Endogenous barriers to entry enhance wlefare in that case. If the valuation of agents for the prize mainly depends on their connections in the network ($v/\beta < 1$), then the sum of payoffs increases with the size of the group dominant network. Endogenous barriers to entry then hurt welfare.

We show in Proposition 3 that a pairwise equilibrium is not efficient if the complete network is not the only pairwise stable network.

Proposition 3. *Every pairwise equilibrium $g^S \neq g^N$ is not efficient. In addition, g^N is not efficient if $g^S \neq g^N$ is a pairwise equilibrium.*

If the valuation of agents for the prize mainly depends on their connections in the network ($v/\beta < 1$), the complete network is the pairwise equilibrium that maximizes the sum of payoffs. Even if the prize is allocated to an agent with the maximal possible valuation in the complete network, it generates less surplus than the star network. In the star network, the gap between the valuation of the center of the star and of the other agents is so important that most of the effort is done by the center of the star. He gets the prize with high probability while the sum of efforts is relatively small. If on the other hand, the valuation of the agents for the prize mainly depends on their fixed valuation for the good ($v/\beta > 1$), the pairwise equilibrium network that generates the highest sum of payoffs is the one with the smallest group of completely connected agents g^S. If g^S is not the complete network, it is not efficient since the sum of payoffs is higher in the adjacent network $g^S + ij$ where the link added involve an agent in the group and another outside the group (see Lemma 4 in the appendix).

In a model of cost-reducing network formation in an oligopolistic market, Westbrock (2010) finds that efficient networks must either be group dominant or have the interlinked star architecture. In the Tullock contest, we show that group dominant networks other than the complete network never maximize total surplus.

5 Conclusion

This paper develops a model of network formation among competitors in a Tullock contest. We establish that the only pairwise equilibrium networks are group dominant networks. Agents in the group are completely connected to each other while those outside the group are not connected and are excluded from the market. Network formation thus acts as a barrier to entry to the contest. We show that whenever networking acts as a barrier to entry, no pairwise equilibrium network is efficient. Endogenous barriers to entry may hurt total surplus because the winner of the prize does not exploit all the possible network benefits. It may also improve surplus because efforts are smaller when competition is less fierce.

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6 Appendix

Lemma 1 establishes that the harmonic mean of the valuation of the x players with the highest valuation, weighted by $(\kappa(g)-x-1/\kappa(g)-x)$ is a function that is increasing in x as long as x is smaller than the number of participating agents. Notice in particular that this function is equal to 0 for $x=1$ and to $1/((1/v_k)+(1/v_l))<v_l$ for $x=2$ where $\phi(k,g)=1$ and $\phi(l,g)=2$. It follows that in every network, there are always at least two agents participating to the contest.

Lemma 1. *For all* $x\in\{1,2,...,\kappa(g)-1\}$*, we have*

$$\frac{(x-1)}{x}h_x(g)<\frac{x}{x+1}h_{x+1}(g) \tag{A1}$$

Proof. Let $x\in\{1,2,...,\kappa(g)-1\}$. Let player l be such that $\phi(l,g)=x+1$ and player k be such that $\phi(k,g)=\kappa(g)$. Then $(A1)$ can rewritten as

$$v_l(g)>\frac{x-1}{x}h_x(g) \tag{A2}$$

The participation constraint of player k may be written as

$$v_k(g)>\frac{\kappa(g)-1-\sum_{i\in N:x+1\leq\phi(i,g)\leq\kappa(g)}v_k(g)/v_i(g)}{x/h_x(g)}.$$

Since $v_k(g)/v_i(g) \leq 1$ for $i \in N$ such that $\phi(i,g) \leq \kappa(g)$, it follows that $v_k(g) > \frac{x-1}{x} h_x(g)$. Since $v_l(g) \geq v_k(g)$, $(A2)$ is satisfied.

□

Proposition 1. Every network $g \subseteq g^N$ such that either $i, j \in K(g)$ and $ij \notin g$ or $d_k(g) > 0$ for some $k \notin K(g)$ is not a pairwise equilibrium.

Proof. Let $g \subseteq g^N$ with $ij \notin g$ for some $i, j \in K(g)$. Suppose first that $d_k(g) > 0$ for some $k \notin K(g)$, then g is not a pairwise equilibrium since player k would be better off by deleting his links. Suppose then that $\kappa(g) = 2$. Since $K(g) = \{i, j\}$ and $ij \notin g$, i and j are connected to other agents because if it was not the case, we would have $v_i(g) = v_j(g) = v$ and $K(g) = N$, a contradiction. Then, a player who has some connections but does not participate is better off by deleting his links. Thus g is not a pairwise equilibrium. In the rest of the proof, we show that the network g such that $\kappa(g) \geq 3$ and $d_k(g) = 0$ for all $k \notin K(g)$ is not a pairwise equilibrium. We decompose the proof into 4 steps. In Step 1, we show that the set of participating agents in the network $g + ij$ is a subset of the set of participating agents in the network g. In Step 2, we show that both players i and j participate in $g + ij$ when the set of participating agents is smaller in $g + ij$ than in g. In Step 3, we show that the players i and j are better off by adding the link ij if the set of participating agents is the same under the networks g and $g + ij$. Finally, in Step 4, we show that if the set of participating agents is smaller under the network $g + ij$ than under g, then there always exists two players k, l that are better off in the network $g + kl$ than in the network g.

Step 1. $K(g + ij) \subseteq K(g)$.

Notice that $h_x(g + ij) \geq h_x(g)$ for all $x \in \{1, 2, ..., n\}$, a property of the comparison between (harmonic) means of two series of numbers $((v_k(g + ij))_{k \in N}$ and $(v_k(g)_{k \in N})$ where the first series has higher numbers than the other. Since $i, j \in K(g)$, $v_k(g) = v_k(g+ij)$ for all $k \notin K(g)$. It follows that the no participation of player k under the network g implies the no participation of player k under the network $g+ij$: $v_k(g) < h_{\kappa(g)}(g)(\kappa(g)-1)/\kappa(g)$ implies $v_k(g+ij) < h_{\kappa(g)}(g+ij)(\kappa(g)-1)/\kappa(g)$ for all $k \notin K(g)$. Thus, $K(g + ij) \subseteq K(g)$.

Step 2. $i, j \in K(g + ij)$.

Using Lemma 1, we know that

$$v_i(g) > \frac{\kappa(g+ij)-1}{\sum_{l\in N:\phi(l,g)\leq\kappa(g+ij)} 1/v_l(g)}$$

Suppose $i, j \notin K(g+ij)$, then $v_k(g+ij) = v_k(g)$ for all k$\in K(g+ij)$. It follows that $v_i(g+ij) > v_i(g) > \frac{\kappa(g+ij)-1}{\sum_{l\in N:\phi(l,g)\leq\kappa(g+ij)} 1/v_l(g)}$, a contradiction. Suppose $i \notin K(g+ij)$ and $j \in K(g+ij)$, then $v_i(g+ij)\sum_{l\in N:\phi(l,g)\leq\kappa(g+ij)} 1/v_l(g+ij) > v_i(g)\sum_{l\in N:\phi(l,g)\leq\kappa(g+ij)} 1/v_l(g) > \kappa(g+ij)-1$, where the first inequality holds since $v_i(g+ij)/v_j(g+ij) > v_i(g)/v_j(g)$ and the second is Lemma 1. Then $i \in K(g+ij)$, a contradiction.

Step 3. $\pi_i(g+ij) > \pi_i(g)$ and $\pi_j(g+ij) > \pi_j(g)$ if $K(g+ij) = K(g)$. We show hereafter that $p_i^*(g+ij)-p_i^*(g) > 0$ both when $v_i(g) > v_j(g)$ and when $v_i(g) \leq v_j(g)$. As a consequence, we conclude that $p_j^*(g+ij)-p_j^*(g) > 0$, and that $\pi_k(g+ij) > \pi_k(g)$ for $k = ij$.

Using $\kappa(g+ij) = \kappa(g) = \kappa$, notice that $p_i^*(g+ij)-p_i^*(g) = [(\kappa-1)/\kappa][(h_\kappa(g)/v_i(g))-h_\kappa(g+ij)/v_i(g+ij)]$.

Thus $p_i^*(g+ij) - p_i^*(g) > 0$ if $h_\kappa(g)/v_i(g) > h_\kappa(g+ij)/v_i(g+ij)$, that is if

$$\sum_{k:\phi(k,g)\leq\kappa} \frac{v_i(g+ij)}{v_k(g+ij)} > \sum_{k:\phi(k,g)\leq\kappa} \frac{v_i(g)}{v_k(g)} \tag{A3}$$

Using $v_k(g+ij) = v_k(g)$ for $k \neq i, j$ and $v_k(g+ij) = v_k(g)+\beta$ for $k = i, j$, and noting that $\sum_{k:\phi(k,g)\leq\kappa(g)} 1/v_k(g+ij) = \sum_{k:\phi(k,g)\leq\kappa(g)}[1/v_k(g)] + [1/v_i(g+ij)] + [1/v_j(g+ij)] - [1/v_i(g)] - [1/v_j(g)]$, let us rewrite Condition (A3) as

$$\beta \sum_{k:\phi(k,g)\leq\kappa} \frac{1}{v_k(g)} - \beta(\frac{1}{v_i(g)} + \frac{1}{v_j(g)}) > \frac{v_i(g)}{v_j(g)} - \frac{v_i(g)+\beta}{v_j(g)+\beta}, \tag{A4}$$

Notice that the left-hand side of Condition (A4) (hereafter LHS) is positive while the right-hand side (RHS) is negative when $v_i(g) \leq v_j(g)$. Let us show that Condition (A4) is also satisfied when $v_i(g) > v_j(g)$. From the participation constraint of player j in the network g, we know that $v_j(g) > (\kappa(g)-1)/(\sum_{k:\phi(k,g)\leq\kappa(g)} \frac{1}{v_k(g)})$. Notice that the RHS is decreasing in $v_j(g)$ while LHS does not depend on $v_j(g)$. We show that Condition (A4) holds even if $v_j(g) = (\kappa(g)-1)/\sum_{k:\phi(k,g)\leq\kappa(g)} \frac{1}{v_k(g)}$. Indeed, we then have $LHS = \beta(\kappa(g)-1)/v_j(g) - \beta/v_i(g) - \beta/v_j(g) > \beta(\kappa(g)-3)/v_j(g) >$

$\beta(v_i(g) - v_j(g))/v_j(g)(v_j(g) + \beta) = RHS$, where the first inequality holds since $v_i(g) > v_j(g)$, while the second holds if

$$(\kappa(g) - 2)v_j(g) + (\kappa(g) - 3)\beta \geq v_i(g). \tag{A5}$$

Suppose that $d_j(g) \geq 1$. Since $ij \notin g$, and $d_k(g) = 0$ for all $k \notin K(g)$, we have $d_i(g) \leq \kappa(g) - 2$. It then follows that $v_i(g) - v_j(g) \leq (\kappa(g) - 3))\beta$ implying that Condition (A5) is satisfied since $\kappa(g) \geq 3$. If on the other hand, $d_j(g) = 0$. It then follows that $\kappa(g) = n$, and Condition (A5) becomes $(n-3)v + (n-3-d_i(g))\beta \geq 0$, which is satisfied when $d_i(g) \leq n-3$. When $d_i(g) = n-2$, the condition becomes

$$(n-3)v \geq \beta. \tag{A6}$$

Notice that $d_k(g) \geq 1$ for each player $k \neq j$ since $d_j(g) = 0$ and $d_i(g) = n-2$. The participation constraint of player j in the network g then implies that $n - 1 < \sum_{k \in N} \frac{v}{v_k(g)} \leq 1 + \frac{v}{v+(n-2)\beta} + \frac{(n-2)v}{v+\beta} < 1 + \frac{(n-1)v}{v+\beta}$. We thus find that $v > (n-2)\beta$, establishing that condition (A6) holds as long as $n \geq 4$. When $N = \{i, j, k\}$ so that $n = 3$, the only network g such that $d_i(g) = n - 2$ and $d_j(g) = 0$ is $g = \{ik\}$. In that case, $p_i^*(g + ij) > p_i^*(g)$ if $v > \beta$, which is satisfied as otherwise player j would not participate in the contest in the network g.

Step 4. If $K(g+ij) \subsetneq K(g)$, then there always exists a pair of players $k, l \in K(g)$ such that $\pi_k(g + kl) > \pi_k(g)$ and $\pi_l(g + kl) > \pi_l(g)$

Step 4.a. Suppose that $\kappa(g + ij) \geq 3$. We show that $\pi_i(g + ij) > x > \pi_i(g)$, where

$$x = v_i(g)(1 - \frac{(\kappa(g+ij) - 1)h_{\kappa(g+ij)}(g)}{\kappa(g+ij)v_i(g)})^2$$

Indeed, from Lemma 1, we have $x > \pi_i(g)$, while from step 2 in this proof, we know that $\pi_i(g + ij) > x$ when $\kappa(g + ij) \geq 3$. Following the same argument, we have $\pi_j(g + ij) > \pi_j(g)$.

Step 4.b. Suppose that $K(g+ij) = \{i, j\}$. Notice that $\kappa(g) - 1 > \min\{d_i(g), d_j(g)\}$ since i and j are not connected to nonparticipating agents, nor among themselves in g. In addition $\min\{d_i(g), d_j(g)\} \geq d_l(g)$ for all $l \in N\backslash\{i, j\}$ since $K(g+ij) = \{i, j\}$. Suppose there exists a pair of players $k, l \in K(g)$, with $kl \notin g$ such that $\kappa(g+kl) \geq 3$. Then, we know that $k, l \in K(g + kl)$ (step 3), and that k and l are better off by adding a link between them (steps 2 or 4.a). If on the other hand $K(g+kl) = \{k, l\}$ for all $k, l \in K(g)$ such that $kl \notin g$, it follows that $min\{d_l(g), d_k(g)\} \geq d_m(g)$ for all $m \in N\backslash\{l, k\}$, for all $k, l \in K(g)$. Then, $d_k(g) = d_l(g)$ for all $k, l \in K(g)$, implying that the payoff of each participating agent $k \in K(g)$ is given by $\pi_k(g) = v_k(g)/\kappa(g)^2$. It then follows that $\pi_i(g + ij) = (v_i(g) + \beta)/4 > \pi_i(g) = v_i(g)/\kappa(g)^2$.

□

Lemma 2 determines the relationships between the value of v/β and the set of participating agents in a group dominant network g^S and in the adjacent networks network $g^S - ij$ and $g^S + ij$. First, it is shown that the isolated agents of the group dominant network g^S participate if and only if the size of the group s is smaller than $(v/\beta)^{1/2}+1$. Second, if a link is removed from the network g^S, the set of participating agents may either be the the set of agents with $s-1$ links, or those with at least $s-2$ links, or the entire population. Finally, if only the members of S participate in the group dominant network g^S, then two isolated agents do not increase sufficiently their valuation by forming a link in order to participate in the contest as long as the size of the group is greater than the threshold $3/2 + \sqrt{5/4 + v/\beta}$.

Lemma 2. *Let $i, j \in S$ and $k, l \notin S$*

(i) $K(g^S) = \begin{cases} S \text{ if } s \geq (v/\beta)^{1/2} + 1 \\ N \text{ if } s < (v/\beta)^{1/2} + 1 \end{cases}$,

(ii) $K(g^S - ij) \in \{S\backslash\{ij\}, S, N\}$

(iii) Suppose $(v/\beta)^{1/2} + 1 \leq s \leq n - 2$ *so that* $K(g^S) = S$*, then*

$$K(g^S + kl) = \begin{cases} S \text{ if } 3/2 + \sqrt{5/4 + v/\beta} \leq s \\ S \cup \{k, l\} \text{ if } s > 3/2 + \sqrt{5/4 + v/\beta} \end{cases}$$

Proof. (i) $K(g^S) = S$ if for $k \notin S$, we have $v_k(g^S) \leq \frac{n-1}{n} h_n(g^S)$, that is if $v \leq \frac{s-1}{s/(v+(s-1)\beta)}$, or $s \geq (\underline{v}/\beta)^{1/2} + 1$. Otherwise if $s < (\underline{v}/\beta)^{1/2} + 1$, then $v_k(g^S) > \frac{n-1}{n} h_n(g^S)$ so that every agent gets a positive payoff by participating.

(ii) For all $m \in N$, we have $v_m(g^S - ij) \in \{v; v + (s-2)\beta; v + (s-1)\beta\}$. Thus, either $K(g^S - ij) = \{m \in N \mid v_m(g^S - ij) \geq v\} = N$ or $K(g^S - ij) = \{m \in N \mid v_m(g^S - ij) \geq v + (s-2)\beta\} = S$, or $K(g^S - ij) = \{m \in N \mid v_m(g^S - ij) \geq v + (s-1)\beta\} = S\backslash\{i, j\}$.

(iii) Given $K(g^S) = S$, we have $K(g^S + kl) = S$ if $v_k(g^S + kl) \leq \frac{s-1}{s} h_s(g^S + kl)$ and $K(g^S + kl) = S \cup \{k, l\}$ otherwise.

□

Proposition 2. A network g is a pairwise equilibrium if

(i) $g = g^N$,

(ii) $g = g^S$ for all $S \subseteq N$ such that $3/2 + \sqrt{5/4 + v/\beta} \leq s \leq n-2$,

(iii) $g = g^{N\backslash\{i\}}$ for all $i \in N$ when either $\frac{(v+\beta)(n-2)}{v+(n-2)\beta} + \frac{(v+\beta)}{(v+(n-1)\beta)} \leq n-2$, or $n \geq (v/\beta)^{1/2} + 2$ and $(v+(n-2)\beta)(1/(n-1))^2 \geq (v+(n-1)\beta)(1 - \frac{(n-1)}{\frac{(n-2)(v+(n-1)\beta)}{v+(n-2)\beta} + 1 + \frac{(v+(n-1)\beta)}{v+\beta}})^2$.

Proof. Let
$$\begin{cases} E_1 = 3/2 + \sqrt{5/4 + v/\beta} \\ E_2 = \frac{(v+\beta)(n-2)}{v+(n-2)\beta} + \frac{(v+\beta)}{(v+(n-1)\beta)} \\ E_3 = (v/\beta)^{1/2} + 2 \\ E_4 = (v+(n-2)\beta)(1/(n-1))^2 \\ E_5 = (v+(n-1)\beta)(1 - \frac{(n-1)}{\frac{(n-2)(v+(n-1)\beta)}{v+(n-2)\beta} + 1 + \frac{(v+(n-1)\beta)}{v+\beta}})^2 \end{cases},$$

Notice that $K(g^S) = N$ for $s < E_3$ by Lemma 2 so that g^S is not pairwise stable by Proposition 1 if $S \neq N$. Also, for all $S \subseteq N$, we have $\pi_i(g^S) > \pi_i(g^S - ij)$ for all $i, j \in S$ since from Lemma 2, either $K(g^S - ij) = S\backslash\{i,j\}$ and $0 = \pi_i(g^S - ij) < \pi_i(g^S) = \frac{v+(s-1)}{s^2}$, or $K(g^S - ij) \in \{S, N\}$ and the result holds by Proposition 1.

(i) g^N is a pairwise equilibrium since $\pi_i(g^N - ij) < \pi_i(g^N)$.

(ii) Let g^S for some $S \subseteq N$ be such that $E_1 \leq s \leq n-2$. By Lemma 2, we have $K(g^S) = S$ since $E_3 \leq s$. Notice that $\pi_k(g^S + kl) = 0$ for $k, l \notin S$ if $v + \beta \leq (s-1)/(\sum_{j \in S} 1/v_j(g))$, while $\pi_k(g^S + ik) = 0$ for $i \in S$, $k \notin S$ if $v + \beta \leq (s-1)/(\sum_{j \in S} 1/v_j(g + ik))$. Thus, a necessary and sufficient condition for g^S to be a pairwise equilibrium when $K(g^S) = S$ is that 2 agents without links in g^S are not better off by adding a link, i.e. $E_1 \leq s$ (see Lemma 2).

(iii) Take $g^{N\backslash\{i\}}$ for some $i \in N$. We show that $g^{N\backslash\{i\}}$ is a pairwise equilibrium if either $K(g^{N\backslash\{i\}} + ij) = N\backslash\{i\}$ or if $K(g^{N\backslash\{i\}}) = N\backslash\{i\}$ and $\pi_j(g^{N\backslash\{i\}} + ij) < \pi_j(g^{N\backslash\{i\}})$. If $E_2 \leq n-2$, we have $K(g^{N\backslash\{i\}} + ij) = N\backslash\{i\}$ so that player i is better of by not adding the link. If $E_2 > n-2$ so that $K(g^{N\backslash\{i\}} + ij) = N$, then two cases should be considered. First suppose that $n \geq E_3$ so that $K(g^{N\backslash\{i\}}) = N\backslash\{i\}$, then $\pi_j(g^{N\backslash\{i\}} + ij) < \pi_j(g^{N\backslash\{i\}})$ if $E_4 \geq E_5$. Second, suppose that $n < E_3$ so that $K(g^{N\backslash\{i\}}) = N$. Then, the network $g^{N\backslash\{i\}}$ is not pairwise stable by Proposition 1.

□

Lemma 3. *$W(g^S) > W(g^T)$ when $s > t$ and $K(g^T) = T$ if and only if $v < \beta$.*

Proof. The total surplus in a group dominant network g^X where $X \subseteq N$ and $K(g^X) = X$ is given by

$$W(g^X) = v/x + ((x-1)/x)\beta.$$

Notice that $K(g^T) = T$ and $t < s$ imply $K(g^S) = S$. Then $W(g^S) > W(g^T)$ when $v < \beta$.

□

Lemma 4. *Suppose $g' = g + ij$, $i \in K(g)$, $j \notin K(g)$ and $K(g+ij) = K(g)$, then $W(g') - W(g) > 0$ if $v_i(g) \geq v_k(g)$ for all $k \in N$.*

Proof. Let $g' = g + ij$, $i \in K(g)$, $j \notin K(g)$ and $K(g+ij) = K(g)$. We then have $W(g') - W(g) = \beta + \frac{(\kappa(g)^2-1)}{\kappa(g)}(h_{\kappa(g)}(g) - h_{\kappa(g)}(g+ij))$

$W(g') - W(g) = \beta + (\kappa(g)^2 - 1)(-\frac{\frac{\beta}{(v_i(g)+\beta)v_i(g)}}{\sum_{k\in K(g)}(1/v_k(g))\sum_{k\in K(g)}(1/v_k(g+ij))})$

$W(g') - W(g) = \beta(1 - \frac{(\kappa(g)^2-1)}{\sum_{k\in K(g)}((v_i(g)+\beta)/v_k(g+ij))\sum_{k\in K(g)}(v_i(g)/v_k(g))})$

Using $v_i(g) \geq v_k(g)$ for all $k \in N$, we have that $\sum_{k\in K(g)}(v_i(g)/v_k(g)) \geq \kappa(g)$ and $\sum_{k\in K(g)}((v_i(g)+\beta)/v_k(g+ij)) \geq \kappa(g)$. It follows that $W(g') - W(g) \geq \beta(1 - \frac{(\kappa(g)^2-1)}{\kappa(g)^2}) > 0$.

□

Proposition 3. Every pairwise equilibrium $g^S \neq g^N$ is not efficient. In addition, g^N is not efficient if $g^S \neq g^N$ is a pairwise equilibrium.

Proof. Suppose first that $v/\beta \leq 1$. Then, simple calculations show that $W(g^*) > W(g^N) > W(g^S)$ for all pairwise equilibrium g^S, where g^* is the star network.[4] If on the other hand, $v/\beta > 1$, then, $W(g^S + ij) > W(g^S) > W(g^N)$ for $i \in S$ and $j \in N\backslash S$. Indeed, $j \notin K(g^S+ij)$ as otherwise g^S would not be a pairwise equilibrium. In addition, $k \in K(g^S + ij)$ for $k \in S\backslash\{i\}$ since the participation constraint of the player k such that $\phi(k,g) = 2$ is always satisfied. Thus $K(g^S + ij) = S$ and Lemma 4 applies. □

[4]Every agent participate both in the complete and in the star network. We have $W(g^*) > W(g^N) \iff v/\beta < n^2 - 3n + 1$.