Constitutions and Social Networks

Ana Mauleon[*] Nils Roehl[†] Vincent Vannetelbosch[‡]

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Abstract

The objective of the paper is to analyze the formation of social networks where individuals are allowed to engage in several groups at the same time. These group structures are interpreted here as social networks. Each group is supposed to have specific rules or constitutions governing which members may join or leave it. Given these constitutions, we consider a social network to be stable if no group is modified any more. We provide requirements on constitutions and players' preferences under which stable social networks are induced for sure. Furthermore, by embedding many-to-many matchings into our setting, we apply our model to job markets with labor unions. To some extent the unions may provide job guarantees and, therefore, have influence on the stability of the job market.

Key words: Social networks, Constitutions, Stability, Many-to-Many Matchings.

JEL classification: C72, C78, D85.

[*]CEREC, Saint-Louis University – Brussels; CORE, University of Louvain, Louvain-la-Neuve, Belgium. E-mail: ana.mauleon@usaintlouis.be

[†]Department of Economics, University of Paderborn, Paderborn; BiGSEM, Bielefeld University, Bielefeld, Germany. E-mail: nils.roehl@uni-paderborn.de

[‡]CORE, University of Louvain, Louvain-la-Neuve; CEREC, Saint-Louis University – Brussels, Belgium. E-mail: vincent.vannetelbosch@uclouvain.be

1 Introduction

There are various situations in economic or daily life where individuals organize themselves in groups, whether for cooperation, coordination, or otherwise. The goal of this paper is to formalize and examine environments where individuals are allowed to engage in several groups at the same time. These group structures are interpreted as social networks in this study.

Depending on the context, formation of these networks occurs for manifold reasons and considering all of them seems to be a virtually impossible venture. In order to be as general as possible, we abstract from activities carried out within each group. That is, we suppose the individuals' preferences to directly depend on the structure of the network. Given these preferences, there might be incentives for joining or leaving certain groups. The salient point is, however, that individuals are not necessarily free to deviate. Some members of a group might have certain property rights which allow them to block the arrival of new members or even give them the power to force existing members to stay. We capture this aspect by introducing the notion of constitution. Each group is supposed to have specific rules governing both which deviations are feasible and who may decide about the deviations. Therefore, the formation of social networks not only depends on the preferences of the individuals but also on the property rights granted by the constitutions.

The framework outlined above captures a wide spectrum of possible applications. A particular one that we are going to discuss in detail is job markets with labor unions. But one could also mention research collaborations, immigration, or social clubs, for instance. These examples already indicate that the rules or constitutions governing which members may join or leave a group may vary greatly. For instance, in some groups it might be possible to dismiss members but in others there might be a protection against this. Or, in some groups entry might be free but in others it might require the consent of other members. Therefore, the constitutional design may have a significant impact on the formation of social networks. Consequently, we are interested in addressing the following questions: What happens in terms of stability if more blocking power is given to the individuals? Under which circumstances is it possible to find constitutions which guarantee the stability of social networks?

The formation of social groups is of fundamental interest and it has been examined from numerous angles. For instance, Ellickson et al. (1999, 2001) as well as Allouch and Wooders (2008) analyze this issue in the context of general equilibrium theory, Acemoglu et al. (2012) provide a dynamic model for studying the stability of societies, and Page and Wooders (2010) formalize club formation as a non-cooperative game, to name but a few. In fact, providing a complete overview over all publications dealing with group formation in a broader sense would exceed the scope of nearly every paper due to the great complexity and diversification of the field. Therefore, the following survey restricts on most closely related branches and outlines which publications particularly influenced our work.

Analyzing group formation but abstracting from activities carried out within each group obviously relates to hedonic coalition formation (e.g., Banerjee et al., 2001; Bogomolnaia and Jackson, 2002). Moreover, studies dealing with economic networks (e.g., Jackson, 2008) or with matching markets (e.g., Roth and Sotomayor, 1990) can also be embedded into our setting. Thus, we contribute indirectly to a stream of literature where the authors combine coalition formation and matching problems (e.g., Cesco, 2012; Pycia, 2012). However, the way we model social networks and preferences is closer to models from matching theory where individuals are not only concerned about which groups they belong to but also about who the other members of the groups are (e.g., Dutta and Masso, 1997; Echenique and Yenmez, 2007; Kominers, 2010).

One of the main contributions of this paper is to formalize constitutional rules within a hedonic setting. This approach is in spirit with some other publications from literature, like Bala and Goyal (2000), Page and Wooders (2009), or Jehiel and Scotchmer (2001), for example. These papers analyze which networks or coalition structures might be expected to emerge under several specific rules governing network or coalition formation, respectively. However, the aforementioned works differ from ours in at least one important aspect. For analyzing which social networks are likely to occur we focus on constitutionally stable networks. A social network is considered to be constitutionally stable if no group of players is modified any more. The salient point is that, in our framework, the stability of a network depends on explicitly modeled constitutions. In the above-mentioned papers, on the contrary, the constitutional rules are varied only implicitly by discussing different stability concepts. For this reason, our approach not only achieves greater generality but it also allows separating more clearly which influence constitutional rules have on group formation.

The analysis conducted in this paper is twofold. On the one hand, we focus on the question whether constitutionally stable networks actually exist and, on the other, we discuss whether they might be reached given that the players apply a "trial-and-error strategy". To this end, we follow Roth and Vande Vate (1990). In the context of marriage problems (or two-sided one-to-one matchings), the authors introduced a Markov process which always results in a stable matching with probability one, even if the individuals act myopically. Later, this work has been extended and varied in several ways (e.g., Chung, 2000; Diamantoudi et al., 2004; Klaus et al., 2010; Kojima and Unver, 2008). In our study, we use basically the same approach but we adopt the terminology of Jackson and Watts (2001, 2002) who examined a similar random process but focused on stochastic stability of economic networks. By means of the notion of improving paths, we formulate requirements on constitutions and preferences guaranteeing that from every social network there always exists an improving path leading to a stable network. It turns out, in fact, that this is equivalent to requiring the existence of a specific version of a common ranking (cf. Banerjee et al., 2001; Farrell and Scotchmer, 1988). We also find that giving more blocking power to the individuals does not necessarily lead to more stability. Indeed, higher blocking power might destroy the existence of the common ranking.

Although the main purpose of this paper is to discuss the formation of social networks in general, the last part is devoted to a particular application, namely to job markets with labor unions. Applying the general results obtained in the sections before allows us to judge, for different levels of unions' strength, whether the job market is likely to become stable or not. In doing so we also find a variation of Roth's "polarization of interests" (cf. Roth, 1984) between employers and employees.

The remainder of the paper proceeds as follows. Section 2 introduces the model and the formal definitions of social networks and constitutions. In Section 3, we discuss conditions for the existence of strongly stable networks. In Section 4, we apply the corresponding results to our model of job markets. Finally, Section 5 contains the conclusions.

2 The Model

Let $N = \{i_1, \ldots, i_n\}$ be a finite set of *players* and let $M = \{c_1, \ldots, c_m\}$ be a finite set of *connections*.

Definition 1. A *social network* h is a mapping $h : M \longrightarrow 2^N$ assigning to each $c \in M$ a subset of players.[1]

A social network h indicates which players are members of which connections. For each $i \in N$ let $M_h(i) = \{c \in M \mid i \in h(c)\}$ be the set of connections player i is a member of. The set of all social networks is denoted by $\mathcal{H}$, and the cardinality of $\mathcal{H}$ is $|\mathcal{H}| = 2^{mn}$. A particular special case is the *empty social network* $h^{\emptyset} \in \mathcal{H}$, with $h^{\emptyset}(c) = \emptyset$ for all $c \in M$. That is, no player is contained in any connection.

Example 1. Suppose there are three players and four connections, i.e., $N = \{i_1, i_2, i_3\}$ and $M = \{c_1, c_2, c_3, c_4\}$. Consider the case where all players are contained

[1] Note that the tuple (N, M, h) is simply a mathematical hypergraph. Therefore, from a technical point of view our definition of social networks also relates to the notions of conference structures (e.g., Myerson, 1980), many-to-many matchings (e.g., Roth, 1984) and social environments (e.g., Fershtman and Persitz, 2012).

in c_1, the players i_2 and i_3 are also contained in c_2 and c_3, while c_4 only contains i_1. This can be described formally by means of the following social network h (see also Figure 1):

$$h(c) = \begin{cases} \{i_1, i_2, i_3\}, & \text{if } c = c_1 \\ \{i_2, i_3\}, & \text{if } c \in \{c_2, c_3\} \\ \{i_1\}, & \text{if } c = c_4. \end{cases}$$

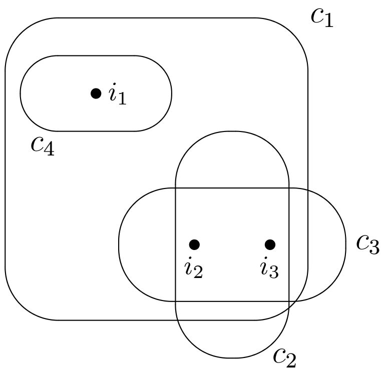

Figure 1: The social network h.

2.1 Constitutions

Each $i \in N$ is supposed to have rational preferences $\succeq^i$ over $\mathcal{H}$. The tuple $\succeq = (\succeq^i)_{i \in N}$ is called a *preference profile*. Depending on the preferences, players might have incentives to alter some connection in a given network. For modeling formally how a connection can be modified we use the symmetric difference $\pm$ defined by $D' \pm D = (D' \setminus D) \cup (D \setminus D')$ for all $D', D \subseteq N$. Correspondingly, given a connection $c \in M$ and a subset of players $D \subseteq N$, let $h \pm (c, D)$ be the social network that is obtained from $h \in \mathcal{H}$ if c is modified by the players in D. More specifically, players in $D \cap h(c)$ leave the connection and players in $D \setminus h(c)$ join it.[2] Formally:

$$(h \pm (c, D))(c') := \begin{cases} h(c) \pm D & \text{if } c = c' \\ h(c') & \text{if } c \neq c' \end{cases} \tag{1}$$

If $D \cap h(c) = \emptyset$, we just write $h + (c, D)$ instead of $h \pm (c, D)$ to stress the fact that no player leaves the connection. If $D \subseteq h(c)$, we just write $h - (c, D)$ instead of $h \pm (c, D)$ to indicate that no player joins the connection.

[2] We use $\pm$ instead of the usual symbol $\triangle$ for denoting the symmetric difference, in order to emphasize that it might be possible that at the same time new members enter a connection while other members leave it.

The central assumption in our model is that each connection could have different rules governing the exit of already existing members and/or the arrival of new members. That is, we introduce the notion of constitution in order to describe both the feasible modifications of a given connection and the coalitions whose support is needed for the modifications to take place. According to these constitutions, some deviations might be precluded, even if all deviating players would benefit from altering the connection.

Definition 2. The *constitution* $\mathcal{C}^c = (\mathcal{C}^c_h)_{h\in\mathcal{H}}$ of connection $c \in M$ is a collection of pairs $\mathcal{C}^c_h = (\mathcal{D}^c_h, S^c_h)$ where (i) $\mathcal{D}^c_h \subseteq 2^N \setminus \{\emptyset\}$ describes the *feasible deviations*, and (ii) for each $D \in \mathcal{D}^c_h$, $S^c_h(D) \subseteq 2^{h(c)}$ specifies a non-empty set of *supporting coalitions.*

That is, for all $c \in M$ and $h \in \mathcal{H}$, $\mathcal{C}^c_h$ consists of two components. The first one, $\mathcal{D}^c_h$, specifies which modifications (with respect to the deviations formalized in (1)) of the connection c are possible. Of course, it might be the case that $\mathcal{D}^c_h = 2^N \setminus \{\emptyset\}$ and then there are no restrictions on feasible deviations. In many applications, however, certain changes of a connection are not possible due to capacity constraints or legal requirements, for example, and this is captured by $\mathcal{D}^c_h$. Moreover, in order to modify c, each deviating group $D \in \mathcal{D}^c_h$ needs the support of at least one *supporting coalition* $S \in S^c_h(D)$. If there exists no such S, the deviation D is blocked. Note that $\emptyset \in S^c_h(D)$ is allowed, too. In this case, the players in $D \in \mathcal{D}^c_h$ do not need the consent of any member of the connection for deviating. Moreover, if $S \in \mathcal{S}^c_h(D)\setminus\{\emptyset\}$, we assume $S' \in \mathcal{S}^c_h(D)$ for all $S' \supseteq S$. That is, if S is a supporting coalition for a certain deviation, all coalitions containing S also have the power to support this deviation. In the following, let $\mathcal{C} := (\mathcal{C}^c)_{c\in M}$. The tuple $(N, M, \succeq, \mathcal{C})$ is called a *society.*

Example 2. Let $N = \{i_1, \ldots, i_n\}$ and $M = \{c_1, c_2, c_3\}$. As an example consider the following three specific constitutions:

(i) If $\mathcal{D}^{c_1}_h = \{D \subseteq N \,|\, |h(c_1) \pm D| \leq 9,\ D \neq \emptyset\}$ and $\mathcal{S}^{c_1}_h(D) = \{S \subseteq h(c_1) \,|\, 2\cdot|S| > |h(c_1)|\}$ for all $h \in \mathcal{H}$ and $D \in \mathcal{D}^{c_1}_h$, the players have to respect a quota of nine and decisions are taken by means of the majority rule.

(ii) Suppose $\mathcal{D}^{c_2}_h = \{D \subseteq N \,|\, l \geq 3 \ \forall\ i_l \in D, D \neq \emptyset\}$ and $\mathcal{S}^{c_2}_h(D) = \{S \subseteq h(c_2) \,|\, h(c_2) \cap D \subseteq S\}$ for all $h \in \mathcal{H}$ and $D \in \mathcal{D}^{c_2}_h$. This reflects the case where deviations require certain qualifications. In this specific example, players need an index of at least three. Moreover, none of the members has property rights for the connection. If a deviation is feasible, the corresponding players have the power to support themselves, i.e., they are free to enter or exit.

(iii) Let $\mathcal{D}_h^{c_3} = 2^N \setminus \{\emptyset\}$ and $\mathcal{S}_h^{c_3}(D) = \{S \subseteq h(c_3) \,|\, i_{\bar{l}} \in S,\ \text{where } \bar{l} \geq l\ \forall\, i_l \in h(c_3)\}$ for all $h \in \mathcal{H}$ and $D \in \mathcal{D}_h^{c_3}$. Here, all deviations are feasible and the player with the highest index acts as a kind of dictator and has perfect property rights. That is, she may decide about both: whether players may join the connection as well as whether they may leave it.

2.2 Stability

For analyzing which social networks might be expected to emerge we propose a notion of stability which requires that no single connection is altered any more.

Definition 3. Given the society $(N, M, \succeq, \mathcal{C})$, a social network h is *constitutionally stable* with respect to the constitutions $\mathcal{C}$ if for all $c \in M$ and $D \in \mathcal{D}_h^c$ we have that: (i) $h \succeq^i h \pm (c, D)$ for at least one $i \in D \setminus h(c)$ or (ii) in each supporting coalition $S \in \mathcal{S}_h^c(D)$ there is a player $j \in S$ with $h \succeq^j h \pm (c, D)$.

Expressed in words, a social network $h \in \mathcal{H}$ is constitutionally stable if and only if for any connection $c \in M$ and any feasible modification $D \in \mathcal{D}_h^c$, at least one of the players joining c does not strictly benefit from deviating or at least one of the members of every supporting coalition $S \in \mathcal{S}_h^c(D)$ is not strictly better off from the deviation.[3] Therefore, we assume that moving from $h \in \mathcal{H}$ to $h \pm (c, D)$ does not necessarily need the consent of players leaving c. That is, some members of the connection might have the power to force other members to leave c even when the excluded players suffer from this exclusion. On the other hand, a player who is not in $c \in M$ cannot be forced to join c. Only if she strictly benefits, she will join it. In the following, let $\mathcal{ST}(\mathcal{C})$ denote the set of constitutionally stable networks with respect to the constitutions $\mathcal{C}$. Moreover, given $h \in \mathcal{H}$, let $\mathcal{A}_h^c(\mathcal{C}) := \{D \in \mathcal{D}_h^c \mid \exists S \in \mathcal{S}_h^c(D) \text{ such that } h \pm (c, D) \succ^i h\ \forall i \in (D \setminus h(c)) \cup S\}$ be the set of all feasible deviations causing an instability in $c \in M$. Notice that if $D \subseteq h(c)$, $D \in \mathcal{D}_h^c$, and $\emptyset \in \mathcal{S}_h^c(D)$, the deviation of D causes an instability by definition although it might be the case that nobody benefits from this modification. In order to exclude exogenous instabilities like these, we will assume $\emptyset \notin \mathcal{S}_h^c(D)$ whenever $D \subseteq h(c)$.

[3] Assuming that players only deviate or support a deviation when their payoffs are strictly bigger has sense when transfers among players are not possible. This is in line with several other stability concepts from literature, like strong stability of Dutta and Mutuswami (1997), pairwise stability of Sotomayor (1999), or core stability of Bogomolnaia and Jackson (2002) and Banerjee et al. (2001), for example.

3 General Results

Generically constitutionally stable social networks might fail to exist and this leads to the question of how the design of constitutions affects the (non-)existence of stable social networks. For approaching this issue, let us start with a straightforward and plausible attempt. Suppose the constitutions grant the players a certain level of blocking power. That is, the members of each connection might have certain property rights allowing them to inhibit modifications of the connection which are not conform to their own preferences.

Remark 1. Let two societies $(N, M, \succeq, \mathcal{C})$ and $(N, M, \succeq, \bar{\mathcal{C}})$ be given and assume $\mathcal{C} \subseteq \bar{\mathcal{C}}$, i.e., $\mathcal{D}_h^c \subseteq \bar{\mathcal{D}}_h^c$ and $\mathcal{S}_h^c(D) \subseteq \bar{\mathcal{S}}_h^c(D)$ for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}_h^c$. Then, $\mathcal{ST}(\bar{\mathcal{C}}) \subseteq \mathcal{ST}(\mathcal{C})$.

The remark follows directly from the definition of constitutional stability. If the sets of feasible deviations and supporting coalitions shrink, the blocking power of each individual player increases and the set of constitutionally stable networks might become larger. However, although the reasoning is very intuitive it might be misleading. In fact, whether more blocking power really implies more stability, strongly depends on the perspective of stability. On the one hand, there might be more stable networks but, on the other hand, reaching them might not be possible any more.

Let us formalize these ideas by adopting the notion of improving paths from Jackson and Watts (2001, 2002). An improving path is a sequence of networks that can emerge when players join or leave some connection based on the improvement the resulting network offers relative to the current network (see Jackson and Watts (2002), p.51). Each network in the sequence differs from the previous one in that one connection is modified by a deviating coalition. This requires that every player joining the connection strictly prefers the resulting network to the current one. Moreover, the deviation should not be blocked and, hence, there should be a supporting coalition that strictly benefits from the deviation.

Definition 4. An *improving path* from $h_0 \in \mathcal{H}$ to $h_k \in \mathcal{H}$ is a sequence of networks $(h_0, h_1, \ldots, h_k)$ such that for all $0 \leq l < k$ there is exactly one $c_l \in M$ with $h_{l+1} = h_l \pm (c_l, D_l)$ for some $D_l \in \mathcal{A}_{h_l}^{c_l}(\mathcal{C})$.

If there exists an improving path from $h \in \mathcal{H}$ to $h' \in \mathcal{H}$, we write $h \mapsto h'$. Moreover, let $I(h) = \{h' \in \mathcal{H} \mid h \mapsto h'\}$ be the set of networks that can be reached by an improving path starting at h.[4] Notice that h is constitutionally stable if and only if $I(h) = \emptyset$.

[4]Note that in improving paths the players are implicitly assumed to care only about the immediate benefit of deviating to the next network but they do not forecast how others might react

A set of networks $H \subseteq \mathcal{H}$ is *closed* if there is no improving path leading out of it, i.e., $I(h) \subseteq H$ for all $h \in H$. Moreover, a set of networks $H \subseteq \mathcal{H}$ with $|H| \geq 2$ is a *cycle* if for any pair $h, h' \in H$, there exists an improving path connecting h to h'.

Lemma 1. *Let the society $(N, M, \succeq, \mathcal{C})$ be given. There exists no closed cycle if and only if, for each network $h \in \mathcal{H}$ that is not constitutionally stable, there exists an improving path leading from this network to a constitutionally stable one.*

Proof. We will show the reverse statement of Lemma 1. If there exists a closed cycle H, by definition there exists no improving path from any $h \in H$ to a constitutionally stable network. This already proofs the first direction. Now suppose there exists a network $h \in \mathcal{H}$ such that there is no constitutionally stable network in $I(h)$. Therefore, this set $I(h)$ must contain at least one cycle H_1. Suppose H_1 is a maximal cycle, i.e., it is not a proper subset of any other cycle. Now, either H_1 is closed and we are done, or it has an improving path going out of it, leading to a new maximal cycle H_2. Note that $H_1 \cap H_2 = \emptyset$. If H_2 is not closed, one can iterate the previous steps and because $I(h)$ is finite, we will finally reach a closed cycle. □

Our Lemma 1 is a modification of Lemma 1 from Jackson and Watts (2002).[5] The non-existence of closed cycles not only implies existence of stable networks but it also guarantees stability in case the agents follow a "trial-and-error" strategy and care only about immediate benefits. In order to make this later point more specific consider the following random process which has been introduced for marriage problems by Roth and Vande Vate (1990). Start with an arbitrary network $h_0 \in \mathcal{H}$. Each round $r \in \mathbb{N}_{\geq 0}$ a pair $(c_r, D_r) \in M \times 2^N$ is drawn randomly with positive probability. If $D_r \in \mathcal{A}_{h_r}^{c_r}(\mathcal{C})$, the process moves to $h_{r+1} := h_r \pm (c_r, D_r)$. Otherwise it remains at $h_{r+1} := h_r$.

Proposition 1. *Let the society $(N, M, \succeq, \mathcal{C})$ be given. The random process described above always (i.e., for all $h_0 \in \mathcal{H}$) converges with probability one to a constitutionally stable network if and only if there are no closed cycles.*

In the context of one-to-one matching problems, the previous result has been established by Roth and Vande Vate (1990) for one-to-one matching problems. Since

to their actions. This approach relates to myopic learning (e.g., Kandori *et al.*, 1993, Kandori and Rob, 1995; Monderer and Shapley, 1996) and is appropriate in relatively complex settings where it is difficult to anticipate all possible changes. In the context of coalition or network formation some authors have relaxed this assumption by analyzing farsighted stability (see, e.g., Herings *et al.*, 2009; Page and Wooders, 2009; Page *et al.*, 2005). Conducting similar studies in our framework is left for future work.

[5] The authors have shown in slightly different terms that it is possible to find "pairwise-stable" networks if there exists no closed cycle.

the reasoning is the same, we omit the proof. But the intuition is straightforward. Since every feasible deviation is drawn with positive probability, also every improving path has a positive probability. Therefore, if for every starting point there is an improving path leading to a constitutionally stable network, the random process converges to one of these networks for sure whenever it is not stopped after finitely many steps. This is particularly remarkable as in our model, network formation is not guided by a social planner or the like. Given the random process introduced above, non-existence of closed cycles is sufficient for guaranteeing that a society induces a constitutionally stable network with probability one even if the players act myopically and the deviations are not organized in a centralized way.

Proposition 2. *Let N, M, and $\succeq$ be given. Let $\mathcal{C} \subseteq \bar{\mathcal{C}}$. Then, non-existence of closed cycles under $\bar{\mathcal{C}}$ does not imply that there are no closed cycles under $\mathcal{C}$.*

Proof. In order to proof the proposition, it is sufficient to construct a suitable example. The one we consider here is a variation of an example from Bogomolnaia and Jackson (2002) and Diamantoudi et al. (2004). There are three players $N = \{i_1.i_2, i_3\}$ and one connection $M = \{c\}$. Thus, $|\mathcal{H}| = 8$. The networks are given by:

	$h_1(c)$	$h_2(c)$	$h_3(c)$	$h_4(c)$	$h_5(c)$	$h_6(c)$	$h_7(c)$	$h^\emptyset$
c	$\{i_1\}$	$\{i_2\}$	$\{i_3\}$	$\{i_1, i_2\}$	$\{i_1, i_3\}$	$\{i_2, i_3\}$	$\{i_1, i_2, i_3\}$	$\emptyset$

and the players' preferences are

$$h_4 \succ^{i_1} h_7 \succ^{i_1} h_5 \succ^{i_1} h_1 \succ^{i_1} h_2 \sim^{i_1} h_3 \sim^{i_1} h_6 \sim^{i_1} h^\emptyset$$
$$h_6 \succ^{i_2} h_7 \succ^{i_2} h_4 \succ^{i_2} h_2 \succ^{i_2} h_1 \sim^{i_2} h_3 \sim^{i_2} h_5 \sim^{i_2} h^\emptyset$$
$$h_5 \succ^{i_3} h_7 \succ^{i_3} h_6 \succ^{i_3} h_3 \succ^{i_1} h_1 \sim^{i_3} h_2 \sim^{i_3} h_4 \sim^{i_3} h^\emptyset.$$

The setting is actually not completely the same as in Bogomolnaia and Jackson (2002), because in their paper the authors study coalition formation (i.e., the set of players is always decomposed into a partition) while we have just one connection containing some of the players. However, "core stability" in their setting corresponds to constitutional stability with respect to the following constitutions $\mathcal{C} = (\mathcal{D}, \mathcal{S})$:

$$\mathcal{D}_h^c = 2^N \setminus \{\emptyset\} \text{ and } \mathcal{S}_h^c(D) = \{S \subseteq h(c) \mid (h(c) \setminus D) \subseteq S, S \neq \emptyset\} \tag{2}$$

for all $h \neq h^\emptyset$. Given $\mathcal{C}$, a priory all modifications of the connection are feasible and a deviation $D \neq h(c)$ takes place if and only if all members of the resulting network benefit from deviating, i.e., $h \pm (c, D) \succ^i h$ for all $i \in h(c) \pm D$. This implies that players who are undesired can be dismissed if the other members agree on this. For the (pathological) special case of $D = h(c)$, it is required that at least one player has to approve the deviation in order to avoid exogenous instabilities. Now, given the constitutions as defined in (2), Diamantoudi et al. (2004) already pointed out that h_7 is the unique constitutionally stable (or "core stable", respectively) network and $H := \{h_4, h_6, h_5\}$ forms a closed cycle. In fact, once H is reached, there is no improving path leading to h_7 because the players act too myopically. However, consider the following constitutions $\bar{\mathcal{C}} = (\bar{\mathcal{D}}, \bar{\mathcal{S}})$. Let $\bar{\mathcal{D}}_h^c = 2^N \setminus \{\emptyset\}$ and

$$\bar{\mathcal{S}}_h^c(D) = \begin{cases} \{S \subseteq h(c) \,|\, (h(c) \setminus D) \subseteq S,\, S \neq \emptyset\} & \text{, if } D \cap h(c) \neq \emptyset \\ \{S \subseteq h(c) \,|\, S \neq \emptyset\} & \text{, if } D \cap h(c) = \emptyset \end{cases}$$

for all $h \neq h^{\emptyset}$. Here, granting access to c just needs the support of only one member of the connection. This obviously implies $\mathcal{C} \subsetneq \bar{\mathcal{C}}$ and, thus, the players have less blocking power (but note that the sets of stable networks coincide). However, in this case, H does not form a closed cycle any more because for all $h \in H$ there is always one member of c who supports deviating from h to h_7. Therefore, given $\bar{\mathcal{C}}$, there exists no closed cycle. □

Proposition 2 dissents Remark 1 in a way. In fact, concluding that more blocking power leads to more stability is too simplistic. Even if the set of constitutionally stable networks becomes larger, it could happen that all improving paths leading to them are severed and closed cycles occur.

Consequently, instead of enhancing the blocking power of the players, it is necessary to find alternative approaches for guaranteeing that the society always induces a constitutionally stable network. To this end, consider once again the example constructed in the proof of Proposition 2. Examining it in detail yields that under $\bar{\mathcal{C}}$ we have $h \mapsto h_7$ for all networks $h \neq h_7$ but $I(h_7) = \emptyset$. Therefore, for all $h \in \mathcal{H}$ there exists a unique element in $I(h)$ which is maximal with respect to "$\mapsto$". On the other hand, this is not true under $\mathcal{C}$ because $H = \{h_4, h_6, h_5\}$ forms a closed cycle and, thus, $I(h) = H$ for all $h \in H$. Although these observations are limited to this specific example, similar considerations also apply in general.

Definition 5. Given the society $(N, M, \succeq, \mathcal{C})$, a *common ranking* $\trianglerighteq$ is a complete and transitive ordering over $\mathcal{H}$ such that $D \in \mathcal{A}_h^c(\mathcal{C})$ implies $h \pm (c, D) \trianglerighteq h$ for all $h \in \mathcal{H}$ and $c \in M$.

A common ranking $\trianglerighteq$ reflects a certain level of consensus between the players. The main idea is that the set of networks can be decomposed into several equivalence classes and once a higher class is reached, this will not be reversed afterwards. Indeed, a deviation takes place only if the joining and supporting players agree that the resulting network is not contained in a lower class than the current one. Note that a priory this is not a restriction at all because it would be possible, for instance, to choose $\trianglerighteq$ in such a way that all networks are equivalent (i.e., $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in \mathcal{H}$). This immediately implies that a (not necessarily unique) common ranking always exists. However, the more consensus about beneficial deviations between the players, the stronger the restrictions that can be imposed by a common ranking.

Proposition 3. *Let the society $(N, M, \succeq, \mathcal{C})$ be given.*

(i) *There are no cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $H \subseteq \mathcal{H}$ there is a unique $\trianglerighteq$–maximal network $\hat{h} \in H$.*

(ii) *There are no closed cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $h \in \mathcal{H}$ there is a unique $\trianglerighteq$–maximal network $\hat{h} \in I(h)$.*

For the proof refer to the appendix. The main importance of Proposition 3 is that it provides an alternative criterion for guaranteeing convergence to a constitutionally stable network. Item (i) states that requiring non-existence of cycles is equivalent to requiring the existence of a special common ranking which identifies a unique maximal element in every subset of networks.[6] Moreover, according to (ii), having this feature only in particular subsets of $\mathcal{H}$ is still strong enough for excluding closed cycles. Therefore, the society induces a constitutionally stable network for sure if and only if the constitutions allow for a common ranking which is sufficiently restrictive. That is, there must be some consent about which feasible deviations are beneficial and which are not.

3.1 Constitutional Rules and Players' Preferences

The remainder of this section is devoted to the analysis of requirements assuring the existence of a common ranking which excludes closed cycles. In order to get more intuition for this, let us consider a stylized example.

Example 3. Suppose there are three players $N = \{i_1, i_2, i_3\}$ and a unique connection $M = \{c\}$. Analogously to the example in the proof of Proposition 2 let $h_3(c) = \{i_3\}$, $h_5(c) = \{i_1, i_3\}$, $h_6(c) = \{i_2, i_3\}$, and $h_7(c) = \{i_1, i_2, i_3\}$. But here, the corresponding feasible deviations are $D^c_{h_3} = D^c_{h_5} = D^c_{h_6} = D^c_{h_7} = \{\{i_1\}, \{i_2\}, \{i_3\}\}$, while the supporting coalitions are given by $\mathcal{S}^c_{h_l}(D) = \{S \subseteq h_l(c) \,|\, i_3 \in S\}$ for all

[6] A common ranking meets this requirement if and only if it is strict. In this case, it is a variation of "Generalized Ordinal Potentials" introduced by Morderer and Shapley (1996). In particular, item (i) of Proposition 3 is closely related to Lemma 2.5 from their publication. Moreover, it also relates to Theorem 1 in Jackson and Watts (2001).

$D \in \mathcal{D}^c_{h_l}$ where $l \in \{3,5,7\}$ and $\mathcal{S}^c_{h_6}(D) = \{S \subseteq h_6(c) \,|\, i_2 \in S\}$ for all $D \in \mathcal{D}^c_{h_6}$. Moreover, the players' preferences are supposed to be as follows:

$$h_7 \succ^{i_1} h_5 \succ^{i_1} h_6 \sim^{i_1} h_3 \succ^{i_1} \dots$$
$$h_7 \succ^{i_2} h_6 \succ^{i_2} h_5 \sim^{i_2} h_3 \succ^{i_2} \dots$$
$$h_6 \succ^{i_3} h_3 \succ^{i_3} h_5 \succ^{i_3} h_7 \succ^{i_3} \dots$$

It is not difficult to check that in this case the set $H = \{h_3, h_6, h_7, h_5\}$ forms a closed cycle because $(h_3, h_6, h_7, h_5, h_3)$ is an improving path (see Figure 2).

Inspecting this cycle in detail we can find a kind of irregularity in the constitutions: In h_3, h_5, and h_7, player i_3 is the only one who may decide about deviations and she even has the power to exclude the other players from the connection. But after allowing i_2 to enter c and moving to h_6, player i_3 looses her strong property rights and i_2 is able to grant i_1 access to the connection. Moreover, not only the constitutions exhibit a kind of irregularity but the players also disagree about the optimal form of the connection. First, as mentioned before, i_3 can exclude i_1 or i_2 in h_7 against their will. If either this exclusion was not possible or the players agreed to being excluded and did not want to join the connection again, the cycle would be splintered. Second, both players, i_2 and i_3, have the power to support a deviation of player i_1. The salient point is that both disagree about whether i_1 should be a member of the connection or not. If there was a common agreement about this, one of the deviations would be blocked.

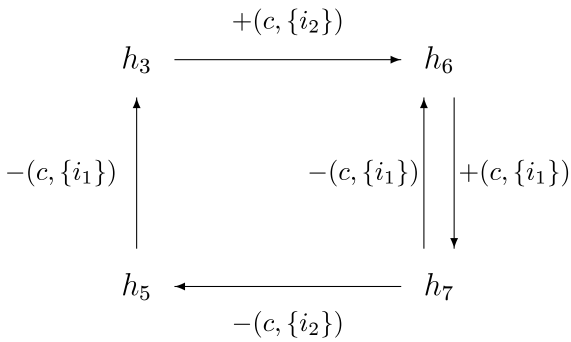

Figure 2: The cycle H.

As the example illustrates, in general there are three main factors which support the occurrence of closed cycles:

(i) constitutions might change strongly even if the network itself does not change much,

(ii) players might be forced to leave a connection against their will, and

(iii) there might be disagreement between the players who decide about the deviations.

In fact, for guaranteeing the existence of a common ranking which satisfies the criterion formalized in Proposition 3(ii), it is necessary to control for all these factors. This implies that we need to find reasonable restrictions on players' preferences and consistency conditions on the constitutions.

Definition 6. Given a closed set $H \subseteq \mathcal{H}$, the constitutions $\mathcal{C} = (\mathcal{C}^c)_{c \in M}$ satisfy

- *regularity* with respect to H if for all $h \in H$ and $c \in M$ we have:

  (i) If $\bar{h}(c) = h(c) \cup \bar{D}$ for some $\bar{h} \in H$ and $\bar{D} \subseteq N \backslash h(c)$, then $\mathcal{D}^c_h = \mathcal{D}^c_{\bar{h}}$ and for all $D \in \mathcal{D}^c_{\bar{h}}$ and $\bar{S} \in \mathcal{S}^c_{\bar{h}}(D)$ there exists $S \in \mathcal{S}^c_h(D)$ with $S \subseteq \bar{S} \subseteq S \cup \bar{D}$.

  (ii) If $D \in \mathcal{D}^c_h$ and $S \in \mathcal{S}^c_h(D)$ with $S \nsubseteq D$, then $h(c) \setminus (S \cup D) \notin \mathcal{S}^c_h(D)$.

- *protection against eviction* with respect to H if for all $h \in H$ and $c \in M$ it holds $D \cap h(c) \subseteq S$ for all $D \in \mathcal{D}^c_h$ and $S \in \mathcal{S}^c_h(D)$;

- *decomposability* with respect to H if for all $h \in H$ and $c \in M$, we have that $D \in \mathcal{D}^c_h$ implies $D' \in \mathcal{D}^c_h$ and $\mathcal{S}^c_h(D) = \mathcal{S}^c_h(D')$ for all $D' \subseteq D$.

The main motivation of *regularity* is to exclude the possibility of skipping back and forth between two networks the whole time: Condition (i) states that the feasible deviations and corresponding supporting coalitions of each $c \in M$ may not vary extremely whenever c changes. If further players are added to the connection, the feasible deviations are supposed to remain the same and supporting coalitions change only as long as they might be complemented by new members. Thus, together with (ii) this implies that if a coalition $S \in \mathcal{S}^c_h(D)$ has the authority to support a deviation $D \in \mathcal{D}^c_h$, this cannot be reversed by another coalition which is neither associated to S nor to D.

If the constitutions satisfy *protection against eviction*, no player can be forced to leave a connection $c \in M$ if she does not want to do it. Modifying c always requires the consent of all deviating players (not only the consent of players who join the connection).

Decomposable constitutions exhibit a kind of independence property. If the deviation of a group of players is feasible, deviations of any subgroup of players are feasible as well and the corresponding supporting coalitions do not change.

Definition 7. A preference profile $\succeq$

- satisfies *self-concern* if $h \sim^i \bar{h}$ for all $i \in N$ and each pair of networks $h, \bar{h} \in \mathcal{H}$ with $M_h(i) = M_{\bar{h}}(i)$ and $h(c) = \bar{h}(c)$ for all $c \in M_h(i)$.

- is *lexicographic* if each agent $i \in N$ has a preference ordering $\hat{\succeq}^i$ over 2^M such that $M_h(i) \hat{\succeq}^i M_{\bar{h}}(i)$ implies $h \succeq^i \bar{h}$ for all $h, \bar{h} \in \mathcal{H}$ with $M_h(i) \neq M_{\bar{h}}(i)$.

- is *uniform* if for all $i \in N$, $c \in M$, and $h, \bar{h} \in \mathcal{H}$ with $i \in h(c) = \bar{h}(c)$, $h - (c, \{k\}) \succ^j h$ implies $\bar{h} - (c, \{k\}) \succ^i \bar{h}$ and $h \succ^j h - (c, \{k\})$ implies $\bar{h} \succ^i \bar{h} - (c, \{k\})$ for $j \in h(c)$, $k \in h(c) \setminus \{i, j\}$.

- is *equable* if for all $i \in N$, $c \in M$, and $h, \bar{h} \in \mathcal{H}$ with $i \in h(c) = \bar{h}(c)$, $h \succ^j h - (c, \{j\})$ for some $j \in h(c)$ implies $\bar{h} \succ^i \bar{h} - (c, \{i\})$ and $h - (c, \{j\}) \succ^j h$ for some $j \in h(c)$ implies $\bar{h} - (c, \{i\}) \succ^i \bar{h}$.

- is *separable* if for all $i \in N$, $c \in M$, and $h, \bar{h} \in \mathcal{H}$ with $i \in h(c) \subseteq \bar{h}(c)$ the two following conditions are satisfied:

  (i) $\bar{h} - (c, D) \succ^i \bar{h}$ if and only if $h - (c, D) \succ^i h$ for all $\emptyset \neq D \subseteq h(c) \setminus \{i\}$.

  (ii) $\bar{h} + (c, D) \succ^i \bar{h}$ if and only if $h + (c, D) \succ^i h$ for all $\emptyset \neq D \subseteq N \setminus \bar{h}(c)$.

Self-concern is a kind of independence property. Player i neither benefits nor suffers if the network changes in such a way that i is not affected directly.

The definition of *lexicographic* preferences is adapted from Dutta and Masso (1997). Under this requirement, player $i \in N$ is mainly concerned about the connections themselves where she is a member of and less about who the other members are. Only if $M_h(i) = M_{\bar{h}}(i)$, might she care about the other players in her connections.

If the preferences of the players are *uniform* and a player leaves a connection, either all remaining members benefit from this deviation or none of them. Note that this is supposed to be independent of the form the other connections have.

Under *equability* player $i \in N$ wants to stay in a connection $c \in M$ only if the other members also want to stay. Suppose, for example, the connections generate a payoff which is distributed equally among the members. Then, if a player has an incentive to leave c, the same goes for i.

Separability as introduced here is a variation of the same-named concept from Banerjee et al. (2001). The idea is that player i's support for a certain leaving or joining group D is independent of the form the connection actually has.

3.2 Non-existence of (Closed) Cycles

Now, combining the restrictions introduced in the previous subsection allows formulating conditions which guarantee non-existence of (closed) cycles and the convergence to a constitutionally stable network.

Proposition 4. *Let a society $(N, M, \succeq, \mathcal{C})$ be given where all constitutions satisfy protection against eviction with respect to a closed set $H \subseteq \mathcal{H}$. If the players' preferences satisfy equability and self-concern, there exist no cycles in H.*

All proofs of this subsection are relegated to the appendix. The requirements of Proposition 4 reflect the three factors identified above which might cause instabilities. Equability and self-concern, for example, impose restrictions on the players' preferences. Both conditions together guarantee that there is only little disagreement about the optimal form of each connection $c \in M$. Moreover, protection against eviction with respect to H has two consequences. On the one hand, as the definition directly implies, players cannot be forced to leave a connection if they do not agree to this. On the other hand, indirectly it also ensures that the constitutions do not change too strongly whenever a connection is altered. More specifically, $S \in \mathcal{S}_h^c(D)$ implies $h(c) \setminus S \notin \mathcal{S}_h^c(D)$ for all $h \in H$, $c \in M$, and $D \in \mathcal{D}_h^c$. The interpretation is similar to regularity. If S has the power to support a deviation of D, this cannot be reversed by other supporting coalitions.

Proposition 5. *Let a society $(N, M, \succeq, \mathcal{C})$ be given where all constitutions satisfy protection against eviction with respect to a closed set $H \subseteq \mathcal{H}$. If the players' preferences are lexicographic, there exist no cycles in H.*

The intuition of the previous result is similar to the intuition of Proposition 4. Obviously, the only difference is that the preferences are not supposed to satisfy equability and self-concern but here they are lexicographic. Therefore, even if there is some disagreement about the optimal form of the connections, it is relegated to a secondary role.

Both previous propositions exclude the existence of not only closed cycles but even of cycles in general. To some extent this is caused by protection against eviction. Indeed, it is not possible to drop or to relax this assumption without reinforcing the requirements on players' preferences.

Proposition 6. *Let a society $(N, M, \succeq, \mathcal{C})$ be given. Assume all constitutions are decomposable and regular with respect to a closed set $H \subseteq \mathcal{H}$. Moreover, suppose the players' preferences are separable, uniform, equable and they satisfy self-concern. Then, there exist no closed cycles in H.*

As the definition directly implies, regularity inhibits the constitutions from varying too extremely and, similar to Proposition 4, equability and self-concern guarantee a certain degree of consent about the optimal form of the network. In addition to this, due to separability and uniformity, in most situations the players are not forced to leave their connections if they do not agree to this. If, for example, some player's entry is supported by a certain coalition, the corresponding members will not change their minds, even if the connection is altered strongly. Thus, the player will only leave again if she has an incentive for deviating.

Note that similar to Proposition 4, it is required that the preferences satisfy equability and self-concern together. Consequently, and as before, it is possible to replace both assumptions in Proposition 6 by lexicography. The intuition is the same: The optimal form of the connections is relegated to a secondary role.

Proposition 7. *Let the society* $(N, M, \succeq, \mathcal{C})$ *be given. Assume all constitutions are decomposable and regular with respect to a closed set* $H \subseteq \mathcal{H}$*. Moreover, suppose the preferences of the players are separable, uniform and lexicographic. Then, there exist no closed cycles in* H*.*[7]

4 Many-to-many Matching Markets

One of the most interesting features of our model is its versatile applicability since overlapping group structures appear in many environments. Consider, for example, many-to-many matching markets. The main primitives of these markets are two finite sets of players E and F, where the members of E are usually interpreted as employees (or workers) and the members of F as firms (see,e.g., Roth, 1984). A (two-sided) many-to-many matching $\mu \subseteq E \times F$ is then simply a collection of worker-firm pairs indicating which employees are working for which firms. Both sides of the market, i.e., all players in E as well as all players in F, are supposed to have preferences over all possible matchings. Thereby, the employees are classically assumed to care only about which firms they work for but not about who their co-workers might be. The owners, on the other hand, are only concerned about the employees working for their firm:

"This involves an assumption that workers are indifferent to who their co-workers might be, and firms are indifferent to whether their employees moonlight at other jobs."

(Roth, 1984, P. 51)

[7] Although the proof proceeds similarly as the one of Proposition 6, the main idea is partially based on Section 5 of Sotomayor (1999).

Many-to-many matching markets can be embedded into our setting in a straightforward way. Let each $c \in M$ represent a firm, i.e., $M := F$. Since in our model the connections do not act as players, we suppose that each firm $c \in M$ has exactly one owner $o_c \in O$. That is, we assume that the set of players $N := E \cup O$ can be decomposed into two (disjoint) subsets, the employees E and the owners O. Given these preliminaries, each matching $\mu \subseteq E \times F$ can be represented by the social network $h^\mu \in H$ which is defined via $h^\mu(c) = \{i \in E \,|\, (i, c) \in \mu\} \cup \{o_c\}$ for all $c \in M = F$. In order to be in line with classical literature on many-to-many matchings, we assume that each owner has no incentive for leaving her firm or for joining any other firm, i.e., we are only interested in the case $O \cap h(c) = \{o_c\}$ for $h \in \mathcal{H}$ and $c \in M$.[8] Nevertheless, since we do not exclude certain network structures a priori, for technical reasons, we also have to define preferences over networks where this requirement is not met. Roth's assumptions on players' preferences imply that each employee $i \in E$ is indifferent among all networks where she is working for the same set of firms, i.e., $h \sim^i \bar{h}$ for all $h, \bar{h} \in \mathcal{H}$ with $M_h(i) = M_{\bar{h}}(i)$. Moreover, given $c \in M$ and $O \cap h(c) = \{o_c\}$, the assumptions also imply $h \sim^{o_c} \bar{h}$ whenever $h(c) = \bar{h}(c)$. For the (pathological) case where $O \cap h(c) \neq \{o_c\}$, we assume $h \pm ((O \cap h(c)) \pm \{o_c\}) \succ^{o_c} h$. Therefore, the preferences of all employees are lexicographic; and restricted to the set $H := \{h \in \mathcal{H} \mid O \cap h(c) = \{o_c\} \; \forall \, c \in M\}$ the same holds for the owners, too.

Since our model is richer than the classical matching approach (in particular, social networks as defined here might be interpreted as one-sided many-to-many matchings), it consequently enables us to model job markets in a more realistically. Complementing this, our formalization of constitutions allows us studying different levels of authority of the owners in a flexible way. For instance, in many countries (especially in Europe) employees are organized in labor unions representing the interests of their members. These unions may guarantee a quite strong protection against dismissal to the workers and, in the short run, the consent of a worker is needed if the owner wants her to leave the firm. Many-to-many matching theory, however, usually concentrates on job markets without strong protection against dismissal like the US job market, for example, and neglects the impact of labor unions. Due to its versatility our model provides an appropriate framework for examining and comparing these different job markets in a convenient way. The remainder of this section is therefore devoted to studying the existence of constitutionally stable networks in three environments that differ in the level of authority that the owners could have.

[8] Thus, we do not consider the possibility of changing the owner. But from a technical point of view it would not be difficult to include this feature into the model.

4.1 Protection against Dismissal

In the following, we will always assume that the employees are allowed to accept as many jobs as they want to. Moreover, the firms have unlimited capacity to hire workers, i.e., given $O \cap h(c) = \{o_c\}$ for $h \in \mathcal{H}$ and all $c \in M$, every possible deviation of the employees is feasible. Nevertheless, quotas could be included easily by allowing only for deviations of subsets which respect a maximal firm size. For sake of completeness, we also have to consider the case where an owner is not part of her firm or other owners are contained in it. Then, we assume that the only feasible deviation is to add the owner and to delete all other owners.

$$\mathcal{D}_h^c = \begin{cases} 2^E, & \text{if } O \cap h(c) = \{o_c\} \\ (O \cap h(c)) \pm \{o_c\}, & \text{if } O \cap h(c) \neq \{o_c\} \end{cases} \tag{3}$$

First we consider the case where unions may guarantee a quite strong protection against dismissal to the workers and the owners do not have the authority to fire them. However, the owner is the only one who may decide about hiring new workers. But each employee is always free to terminate her job if she has an incentive to do it. These considerations lead to the following set of supporting coalitions:

$$\mathcal{S}_h^c(D) = \begin{cases} \{S \subseteq h(c) \mid D \cap h(c) \subseteq S \text{ and } o_c \in S\}, & \text{if } O \cap h(c) = \{o_c\} \text{ and } D \nsubseteq h(c) \\ \{S \subseteq h(c) \mid D \subseteq S\}, & \text{if } O \cap h(c) = \{o_c\} \text{ and } D \subseteq h(c) \\ \{\emptyset\}, & \text{if } O \cap h(c) \neq \{o_c\} \end{cases}$$

Note that for the case of $O \cap h(c) \neq \{o_c\}$, we assume that the empty set is the only supporting coalition and, thus, these networks are not stable by construction.

Corollary 1. *There are no cycles in "Protection against Dismissal".*

Proof. This follows immediately from Proposition 5 because the players' preferences are lexicographic and we also have protection against dismissal with respect to the closed set H. □

At first sight, the previous result might be slightly surprising because in many studies about two-sided many-to-many matchings the existence of stable matchings is an issue (e.g., Sotomayor, 2004). This is mainly due to the fact that this literature examines environments where the owners are free to fire a worker if they benefit from it. Indeed, protection against dismissal is the driving force of the previous result. Let $\mathcal{ST}^{\text{PD}}$ denote the set of stable networks in Protection against Dismissal. Notice that this set contains the worker-optimal networks which are defined as follows. Suppose $\bar{M}^i \subseteq M$ is a set of firms which is mostly preferred by player $i \in E$. Then, if h^{wo} is given by $h^{wo}(c) = \{i \in E \mid c \in \bar{M}^i\} \cup \{o_c\}$ for all $c \in M$, every employee is assigned to a set of firms she preferably wants to work for and, thus, she obviously has no incentive for deviating.

4.2 Hire and Fire

Let us now focus on job markets without strong protection against dismissal. Translated to the model considered here, this means that the owners have the right to fire workers even if these do not agree to leaving. This aspect can be captured by considering the following supporting coalitions:

$$\mathcal{S}_h^c(D) = \begin{cases} \{S \subseteq h(c) \mid o_c \in S\}, & \text{if } O \cap h(c) = \{o_c\},\ D \nsubseteq h(c) \\ \{S \subseteq h(c) \mid D \subseteq S \text{ or } o_c \in S\}, & \text{if } O \cap h(c) = \{o_c\},\ D \subseteq h(c) \\ \{\emptyset\}, & \text{if } O \cap h(c) \neq \{o_c\} \end{cases}$$

Let $\mathcal{ST}^{\text{HF}}$ be the set of stable networks in "Hire and Fire". Note that Remark 1 implies $\mathcal{ST}^{\text{HF}} \subseteq \mathcal{ST}^{\text{PD}}$. However, it is well known that without further assumptions the existence of stable networks in Hire and Fire is not assured (as can be easily seen by means of an example with two workers and two firms). Thus, in order to exclude existence of closed cycles it is necessary to impose further restrictions on constitutions or preferences. For instance, we could proceed similarly as in Proposition 7 since the preferences of the employees are lexicographic and the constitutions in Hire and Fire are not only decomposable but also regular with respect to H. However, it is not necessary to impose such strong assumptions as in Proposition 7. Since the owners are the only players who have decision making power and because they never leave their firm, uniformity is not needed and it is sufficient to additionally assume that the owners' preferences are separable.

Proposition 8. *If the preferences of the owners are separable, there exists no closed cycle in Hire and Fire.*

Remark 2. This proposition is in line with several other well-known publications from the literature, like the papers from Roth and Vande Vate (1990), Chung (2000), Diamantoudi et al. (2004), and especially Kojima and Unver (2008). Similar to our result, Kojima and Unver (2008) have shown in the context of two-sided many-to-many matchings that if workers and owners have, respectively, "substitutable" (see Roth, 1984) and "responsive" (see Roth, 1985) preferences, then there always exists an improving path leading to a stable matching. In fact, the assumptions we impose in Proposition 8 are less restrictive. Given the preferences defined at the beginning of this section and if only deviations of single players are feasible, responsiveness of the owners' preferences implies separability which in turn implies substitutability (converse implications do not hold). Therefore, Proposition 8 complements their findings.

Although we have $\mathcal{ST}^{\text{HF}} \subseteq \mathcal{ST}^{\text{PD}}$, the converse inclusion does not necessarily hold. Therefore, there might exist networks which are stable in Protection against Dismissal that would not be stable if the owners' level of authority is sufficiently high. In particular, due to the characteristics of Hire and Fire, if $h \in \mathcal{ST}^{\text{PD}}$ but $h \notin \mathcal{ST}^{\text{HF}}$, there is at least one owner who would like to fire some of her employees against their will. This already indicates that the interests of both sides of the market might be opposed in a way. For deepening these considerations further we need to enhance separability.

Definition 8. A preference profile $\succeq$ is strongly separable if for all $i \in N$, $c \in M$, and $h, \bar{h} \in \mathcal{H}$ with $i \in h(c) \subseteq \bar{h}(c)$, the two following conditions are satisfied:

(i) $\bar{h} - (c, D) \succ^i \bar{h}$ if and only if $h - (c, D) \succ^i h$ for all $\emptyset \neq D \subseteq h(c)$.

(ii) $\bar{h} + (c, D) \succ^i \bar{h}$ if and only if $h + (c, D) \succ^i h$ for all $\emptyset \neq D \subseteq N \setminus \bar{h}(c)$.

As the name implies, strong separability is a stronger requirement than separability. Again, player i's support for a certain leaving or joining group is independent of the other members of the connection. But, under strong separability, this is also true if i belongs to the deviating group, i.e., if i leaves the connection. Translated to Hire and Fire, this basically means that i's preference about whether to work for a firm $c \in M$ or not is independent of the other firms she is working for.

Proposition 9. *Assume the workers' preferences are strongly separable and the owners' preferences are separable. Moreover, suppose the worker-optimal network h^{wo} is uniquely determined. Then, $h^{wo} \in \mathcal{ST}^{HF}$ if and only if $\mathcal{ST}^{PD} = \mathcal{ST}^{HF}$.*

Proof. If $\mathcal{ST}^{\text{PD}} = \mathcal{ST}^{\text{HF}}$, then also $h^{wo} \in \mathcal{ST}^{\text{HF}}$ because h^{wo} is always stable in Protection against Dismissal and there remains nothing to show. For the other direction, suppose the statement is not true, i.e., $h^{wo} \in \mathcal{ST}^{\text{HF}}$ but $\mathcal{ST}^{\text{HF}} \subsetneq \mathcal{ST}^{\text{PD}}$. Let $\bar{h} \in \mathcal{ST}^{\text{PD}} \setminus \mathcal{ST}^{\text{HF}}$. Then, there must be an owner o_c who would block $\bar{h}$ if her property rights are strong enough, i.e., there exists an employee $i \in \bar{h}(c)$ such that $\bar{h} - (c, \{i\}) \succ^{o_c} \bar{h}$. Because o_c's preferences are separable and h^{wo} is stable, this implies $i \notin h^{wo}(c)$. Otherwise, the owner would also have an incentive to dismiss the employee in h^{wo}. Thus, uniqueness of h^{wo} yields that i strictly prefers h^{wo} to $h^{wo} + (c, \{i\})$. In particular, because her preferences are supposed to be strongly separable, she would also have a strict incentive for leaving connection c at $\bar{h}$, but this contradicts the stability of this network. □

Proposition 9 is in line with Roth (1984). Under the requirement that the preferences of the owners and employees are substitutable, the author finds a "conflict of interest between agents on opposite sides [of the market]" (Roth (1984), p.47). A similar conflict also arises here: Given (strong) separability of the players' preferences, the stable outcome which would be blocked first by the owners is the worker-optimal network. Indeed, this fact is completely independent of specific working conditions such as wages or the working environment, for example, because we abstract from factors like these. Moreover, as will be shown in the following, the conflict becomes even stronger if the owners' level of authority is raised higher.

4.3 Slavery

Roughly speaking, "Slavery" is the counterposition of Protection against Dismissal. Here, the owners not only have the power to decide about new employees but also about whether workers may leave their firm or not:

$$\mathcal{S}_h^c(D) = \begin{cases} \{S \subseteq h(c) \mid o_c \in T\}, & \text{if } O \cap h(c) = \{o_c\} \text{ and } D \nsubseteq h(c) \\ \{S \subseteq h(c) \mid o_c \in T\}, & \text{if } O \cap h(c) = \{o_c\} \text{ and } D \subseteq h(c) \\ \{\emptyset\}, & \text{if } O \cap h(c) \neq \{o_c\} \end{cases}$$

By applying Proposition 5 we get the following result:

Corollary 2. *Every improving path in Slavery leads to a constitutionally stable network.*

Let $\mathcal{ST}^{\text{SL}}$ be the corresponding set of stable networks.

Remark 3. It is easy to check that a network is stable in Hire and Fire if and only if it is stable in Protection against Dismissal and Slavery, i.e., $\mathcal{ST}^{\text{HF}} = \mathcal{ST}^{\text{PD}} \cap \mathcal{ST}^{\text{SL}}$. But it might be the case that the intersection of the sets of stable networks is empty. However, according to Corollary 1 and Corollary 2 there exist no cycles in Protection against Dismissal and Slavery. Therefore, a simple algorithm for finding stable networks in Hire and Fire (in case they exist) is to determine the sets of maximal elements of all improving paths in the two other settings and to check whether the intersection of these sets is non-empty.

Analogously to worker-optimal networks it is also possible to define firm-optimal networks. Let $\hat{E}^c \subseteq E$ be a set of employees which is mostly preferred by player o_c and define h^{fo} by $h^{fo}(c) = \hat{E}^c \cup \{o_c\}$ for all $c \in M$. Then, none of the owners has an incentive for deviating and, thus, the network is stable in Slavery.

Proposition 10. *Assume the workers' preferences are strongly separable and the owners' preferences are separable. Moreover, suppose the firm-optimal network h^{fo} is uniquely determined. Then, $h^{fo} \in \mathcal{ST}^{HF}$ if and only if $\mathcal{ST}^{SL} = \mathcal{ST}^{HF}$.*

Proof. Because Slavery is symmetric to Protection against Dismissal, the proof proceeds analogously to the one of Proposition 9, just by reversing the role of owners and employees. □

Proposition 10 has two implications. First, it shows that the owners can enforce the network which is most beneficial for them if they have a high level of authority. Second, this network would be the first network which is rejected by the employees. In fact, this result extends and reinforces the interpretation of Proposition 9 in a straightforward way: Each side of the market would be worse off if the other side obtains more property rights. If, for example, labor unions narrow the owners' level of authority, the employees would benefit from this and vice versa. Recall that this insight is independent of further working conditions (like wages), which we do not consider explicitly in our model. In particular, this implies that Roth's "polarization of interests" seems to achieve great generality.

5 Conclusion

Even though there is an immense and rich body of literature on the stability of networks (or group structures, respectively), in most of these studies, the stability concepts the authors use are relatively rigid since they do not consider explicitly the rules governing network formation. Indeed, the most distinctive feature of our framework is the formal introduction of constitutions which enable us modeling these rules in a very flexible way. Using this approach we find that enhancing the blocking power of the players does not necessarily lead to more stability. Moreover, we show that the society induces a constitutionally stable network if and only if there is a certain degree of consent between the players about which feasible deviations (according to the constitutions) are beneficial and which are not. In this context, we also discuss conditions under which this criterion is satisfied. By applying our model to job markets with labor unions we find a variation of Roth's "polarization of interests": The workers generically suffer if the degree of authority of the owners is raised and vice versa. In addition to this, we also show that the markets always become stable if the property rights of one side of the market become sufficiently strong.

Although the model we analyze in this paper expands well-established branches like Network Theory or Matching Theory, for example, it is still subject to certain limitations which narrow the field of possible applications. For instance, assuming myopic behavior is reasonable for a start, but it is well-justified only in complex settings where it is extremely difficult to anticipate all possible deviations. Therefore, it might be worth analyzing which results could be obtained if players act farsightedly. Another natural extension is to examine situations where it is possible to add new players or connections to the society. To incorporate this kind of dynamics, it would be necessary to relax the assumption of fixed sets of players and connections. Furthermore, under certain requirements, common rankings relate to ordinal potentials. Since there are numerous publications on potential functions (e.g., Hart and Mas-Colell, 1989; Monderer and Shapley, 1996; Page and Wooders, 2010; Qin, 1996; Slikker, 2001), it seems interesting to study whether the corresponding results also extend to the model introduced here.

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A Appendix: Proofs

Proof of Proposition 3(i)

In order to show that existence of $\trianglerighteq$ implies the non-existence of cycles, we will consider the counterposition of this statement. Therefore, assume there is a cycle $H \subseteq \mathcal{H}$. Since there exists a path from each network in H to every other network in H, if $\trianglerighteq$ is a common ranking, we must have $\bar{h} \trianglerighteq \breve{h}$ as well as $\breve{h} \trianglerighteq \bar{h}$ for all $\bar{h}, \breve{h} \in H$. Thus, there is no unique $\trianglerighteq$-maximal element in H.

For the other direction suppose there exists no cycle. The following algorithm proceeds in a similar way as the one in the proof of Theorem 1 in Jackson and Watts (2001). We start with the binary relation $\trianglerighteq_1$ where $h \triangleright_1 \bar{h}$ if and only if there exists an improving path from $\bar{h}$ to h. Because there is no cycle, $\trianglerighteq_1$ is strict. Moreover, for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}^c_h$, deviating from h to $h \pm (c, D)$ always implies $h \pm (c, D) \triangleright_1 h$ by construction. However, $\trianglerighteq_1$ is not necessarily complete. Let $\tilde{h}, \breve{h} \in \mathcal{H}$ with neither $\tilde{h} \triangleright_1 \breve{h}$ nor $\breve{h} \triangleright_1 \tilde{h}$. We construct $\bar{\trianglerighteq}_1$ by adding $\tilde{h} \bar{\triangleright}_1 \breve{h}$ to

$\trianglerighteq_1$, i.e., $h \mathrel{\bar{\trianglerighteq}_1} \bar{h}$ if and only if $h \trianglerighteq_1 \bar{h}$ or $h = \tilde{h}$ and $\bar{h} = \breve{h}$. Moreover, let $\trianglerighteq_2$ be the transitive closure of $\bar{\trianglerighteq}_1$. We will show that $\trianglerighteq_2$ still represents the preference profile of the players, i.e., deviating from h to $h \pm (c, D)$ always implies $h \pm (c, D) \triangleright_2 h$ for all $c \in M$ and $D \in \mathcal{D}^c_h$. Suppose this is not true, that is, suppose there exist $h' \in \mathcal{H}$, $c \in M$, $D \in \mathcal{D}^c_{h'}$, and $S \in \mathcal{S}^c_{h'}$ with $h'^i\ h'$ for all $i \in (D \setminus h'(c)) \cup S$ but $h' \trianglerighteq_2 h \pm (c, D)$. Thus, there exists a sequence of networks $(h_0, h_1, \ldots, h_k)$ with $h_0 = h'$, $h_k = h' \pm (c, D)$ and $h_0 \mathrel{\bar{\trianglerighteq}_1} h_1 \mathrel{\bar{\trianglerighteq}_1} \ldots \mathrel{\bar{\trianglerighteq}_1} h_k$. Assume the sequence is of minimal length. This implies that $h_l = h_{l'}$ only if $l = l'$ for all $l, l' \in \{0, 1, \ldots, k\}$. Suppose there exists an $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\breve{h}, \tilde{h}\}$. Because $h_{l'} \neq \breve{h}, \tilde{h}$ for all $l' \notin \{l-1, l\}$ this yields

$$h_l \trianglerighteq_1 h_{l+1} \trianglerighteq_1 \ldots \trianglerighteq_1 h_k = h' \pm (c, D) \trianglerighteq_1 h' = h_0 \trianglerighteq_1 \ldots \trianglerighteq_1 h_{l-1}$$

and, thus, there exists an improving path from $\tilde{h}$ to $\breve{h}$ or vice versa. This contradicts the assumption that the two networks are not comparable under $\trianglerighteq_1$. Therefore, there exists no $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\breve{h}, \tilde{h}\}$. From this follows $h_0 \trianglerighteq_1 h_1 \trianglerighteq_1 \ldots \trianglerighteq_1 h_k$ which contradicts the assumption that there is no cycle. Thus, $\trianglerighteq_2$ still represents the preferences of the players and by construction it is also transitive and strict. If it is not complete, the previous steps can be iterated. Because the set of networks is finite, the iteration will stop after finitely many steps and we obtain a common ranking $\trianglerighteq$ which is strict. In particular, strictness implies that for each $H \subseteq \mathcal{H}$ there is a unique $\trianglerighteq$-maximal network $\hat{h} \in H$. □

Proof of Proposition 3(ii)

The first direction proceeds analogously to the first direction of Part (i). Let a common ranking $\trianglerighteq$ and a set of networks $H \subseteq \mathcal{H}$ be given. If H forms a closed cycle, we have $I(h) = I(h') = H$ and $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in H$. But this would contradict that there is a unique $\trianglerighteq$-maximal network in H and, thus, there cannot exist a closed cycle.

For the other direction suppose there exist no closed cycles. The first step of the construction of the common ranking proceeds in the same way as the one of Part (i). That is, we start with $\trianglerighteq_1$ where $h \trianglerighteq_1 \bar{h}$ if and only if there exists an improving path from $\bar{h}$ to h. But note that here this binary relation is not necessarily strict. Since by assumption there are no closed cycles, there exists at least one constitutionally stable network $h' \in \mathcal{H}$. If this network is uniquely determined, according to Lemma 1 it is contained in every closed subset $H \subseteq \mathcal{H}$ and $\trianglerighteq_1$ can then obviously be extended to a complete ranking where h' is the unique maximal element. Therefore, in the following, suppose there exists a further constitutionally stable network $h'' \in \mathcal{H}$. In particular, this implies that neither $h' \trianglerighteq_1 h''$ nor $h'' \trianglerighteq_1 h'$. Let $\tilde{h}, \breve{h} \in \mathcal{H}$ be an arbitrary pair of networks not comparable under $\trianglerighteq_1$. Analogously to above, $\bar{\trianglerighteq}_1$ is constructed by adding $\tilde{h} \bar{\triangleright}_1 \breve{h}$ to $\trianglerighteq_1$, i.e., $h \bar{\trianglerighteq}_1 \bar{h}$ if and only if $h \trianglerighteq_1 \bar{h}$ or $h = \tilde{h}$ and $\bar{h} = \breve{h}$. Again, let $\trianglerighteq_2$ be the transitive closure of $\bar{\trianglerighteq}_1$. Note that by construction $h' \trianglerighteq_2 h''$ would imply $h' \ntrianglerighteq_2 h''$ and vice versa. If $\trianglerighteq_2$ is not complete, because of finiteness of $\mathcal{H}$ we can iterate the previous steps until a complete ranking $\trianglerighteq$ is reached. We will show that h' and h'' are still not equivalent under $\trianglerighteq$. This, in fact, has the following implication: If $\breve{h}$ is $\trianglerighteq$–maximal in a closed subset $H \subseteq \mathcal{H}$, it has to be constitutionally stable by construction and w.l.o.g. we may assume $\breve{h} = h'$. Then, for any other stable network $h'' \in H$, we must have $h' \triangleright h''$ and, thus, h' is the unique $\trianglerighteq$–maximal element in H.

In order to show h' and h'' are still not equivalent under $\trianglerighteq$, let $\trianglerighteq_k$ be the binary relation constructed in the k-th step of the algorithm described in the previous passage. For $k = 1, 2$ we already know that $h' \trianglerighteq_k h''$ would imply $h' \ntrianglerighteq_k h''$ and vice versa. We will show inductively that this is also satisfied for all other k. Therefore, let $k \geq 3$ and suppose that h' and h'' are still not equivalent under $\trianglerighteq_{k-1}$. Moreover, assume this is not satisfied under $\trianglerighteq_k$, i.e., we have $h' \trianglerighteq_k h''$ as well as $h'' \trianglerighteq_k h'$. This assumption will lead to a contradiction. Let $\tilde{h}^{(k-1)}, \breve{h}^{(k-1)} \in \mathcal{H}$ be the corresponding pair of networks not comparable under $\trianglerighteq_{k-1}$ which is added in the next step. We will distinguish three cases:

Case 1: $h' \triangleright_{k-1} h''$.

Because we assume h' and h'' are not equivalent under $\trianglerighteq_{k-1}$, this implies that there exists a sequence of networks $(h_1, \ldots, h_l)$ with $h_1 = h''$, $h_l = h'$, and $h_1 \bar{\trianglerighteq}_{k-1} \ldots \bar{\trianglerighteq}_{k-1} h_l$. Moreover, from this also follows that there exists $1 \leq l' \leq l - 1$ with $\{h_{l'}, h_{l'+1}\} = \{\tilde{h}^{(k-1)}, \breve{h}^{(k-1)}\}$. But then

$$h_{l'+1} \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} h' \triangleright_{k-1} h'' \trianglerighteq_{k-1} \ldots \triangleright_{k-1} h_{l'},$$

which contradicts that $\tilde{h}^{(k-1)}$ and $\breve{h}^{(k-1)}$ are not comparable under $\triangleright_{k-1}$.

Case 2: $h'' \triangleright_{k-1} h'$.

This case proceeds analogously to the previous one by just reversing the roles of h' and h''.

Case 3: h' and h'' are not comparable under $\trianglerighteq_{k-1}$.

If h' and h'' are equivalent under $\trianglerighteq_k$ but not under $\trianglerighteq_{k-1}$, there must be two sequences of networks $(h_1, \ldots, h_l)$ and $(\bar{h}_1, \ldots, \bar{h}_{\bar{l}})$ with $h_1 = \bar{h}_{\bar{l}} = h'$, $h_l = \bar{h}_1 = h''$, and

$$h_1 \bar{\trianglerighteq}_{k-1} \ldots \bar{\trianglerighteq}_{k-1} h_l = \bar{h}_1 \bar{\trianglerighteq}_{k-1} \ldots \bar{\trianglerighteq}_{k-1} \bar{h}_{\bar{l}}.$$

Moreover, there exist $1 \leq l' \leq l-1$ and $1 \leq \bar{l}' \leq \bar{l}-1$ with $\{h_{l'}, h_{l'+1}\} = \{\bar{h}_{\bar{l}'}, \bar{h}_{\bar{l}'+1}\} = \{\tilde{h}^{(k-1)}, \breve{h}^{(k-1)}\}$. In particular, this yields

$$h_{l'} \bar{\trianglerighteq}_{k-1} h_{l'+1} \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} h'' \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} \bar{h}_{\bar{l}'} \bar{\trianglerighteq}_{k-1} \bar{h}_{\bar{l}'+1}$$

which could only be satisfied if $\tilde{h}^{(k-1)}$ and $\breve{h}^{(k-1)}$ are comparable under $\trianglerighteq_{k-1}$.
□

Proof of Proposition 4

Let $(h_0, \ldots, h_k)$ with $h_0, \ldots, h_k \in H$ be an improving path. Moreover, suppose $h_0 = h_k$, that is, suppose $\{h_0, \ldots, h_k\}$ forms a cycle. By construction of improving paths there exists $c_0 \in M$ and $D_0 \in \mathcal{D}_{h_0}^{c_0}$ with $h_1 = h_0 \pm (c_0, D_0)$.

Case 1: $D_0 \nsubseteq h_0(c_0)$, i.e., there exists $i_0 \in D_0 \setminus h_0(c_0)$.
Thus, $h_1 \succ^{i_0} h_0$. Because all players are self-concerned this implies

$$h_1 \succ^{i_0} h_0 \sim^{i_0} h_0 \pm (c_0, D_0 \setminus \{i_0\}) = h_1 - (c_0, \{i_0\}).$$

In other words, after joining the connection player i_0 has no incentive to leave it unilaterally. By equability this is true for all other $i \in h_1(c_0)$. Moreover, let $D \in \mathcal{D}_{h_1}^{c_0}$ with $D \cap h_1(c_0) \neq \emptyset$ and let $i \in D \cap h_1(c_0)$. Then:

$$h_1 \succ^{i} h_1 - (c_0, \{i\}) \sim^{i} (h_1 - (c_0, \{i\})) \pm (c_0, D \setminus \{i_0\}) = h_1 \pm (c_0, D).$$

Because the constitutions satisfy protection against eviction by assumption, no player can be forced to leave a connection against her will. Thus, all players in $h_1(c_0) \cap D$ would block the deviation from h_1 to $h_1 \pm (c_0, D)$. We will show next that the same is also true in h_2. To this end, let $c_1 \in M$ and $D_1 \in \mathcal{D}_{h_1}^{c_1}$ with $h_2 = h_1 \pm (c_1, D_1)$. If $c_1 = c_0$, the previous discussion implies $D_1 \cap h_1(c_0) = \emptyset$ and, by similar arguments as before, it can be shown that $h_2 \succ^{i} h_2 \pm (c_0, D)$ for all $i \in h_2(c_0)$ and $D \in \mathcal{D}_{h_2}^{c_0}$ with $i \in D$. However, if $c_1 \neq c_0$, then $h_2(c_0) = h_1(c_0)$. Thus, by equability $h_2 \succ^{i} h_2 \pm (c_0, D)$ for all $i \in h_2(c_0)$ and $D \in \mathcal{D}_{h_2}^{c_0}$ with $i \in D$. Iterating these arguments implies $h_l \succ^{i} h_l - (c_0, D)$ for all $1 \leq l \leq k$, $i \in h_l(c_0)$ and $D \in \mathcal{D}_{h_l}^{c_0}$ with $i \in D$. In particular, if $h_0 = h_k$, then $h_0 = h_k \succ^{i_0} h_k - (c_0, D_0) = h_1$ and, thus, i_0 would have blocked deviating to the network h_1.

Case 2: $D_0 \subseteq h_0(c_0)$, i.e., $h_1 = h_0 - (c_0, D_0)$.
Thus, $h_1(c_0) \subsetneq h_0(c_0)$ and, moreover, $h_0 - (c_0, D_0) \succ^{i_0} h_0$ by protection against eviction. Let $i_0 \in D_0$. Because $h_0 = h_k$, there must be $1 \leq k' \leq k-1$ and

$D \in \mathcal{D}^{c_0}_{h_{k'}}$ with $h_{k'+1} = h_{k'} \pm (c_0, D)$ and $i_0 \in D$. In particular, $h_{k'+1} \succ^{i_0} h_{k'}$. Similar to Case 1, exploiting that all players are self-concerned yields

$$h_{k'+1} \succ^{i_0} h_{k'} \sim^{i_0} h_{k'} \pm (c_0, D \setminus \{i_0\}) = h_{k'} - (c_0, \{i_0\}).$$

Therefore, equability implies $h_{k'+1} \succ^{i} h_{k'+1} - (c_0, \{i\})$ for all $i \in h_{k'+1}(c_0)$. Now, by advancing analog arguments as in Case 1 it is possible to show that this also yields $h_l \succ^i h_l - (c_0, D)$ for all $k'+1 \leq l \leq k$, $i \in h_l(c_0)$ and $D \in \mathcal{D}^{c_0}_{h_l}$ with $i \in D$. In particular, this is also true for $h_0 = h_k$. But this contradicts again $h_0 - (c_0, D_0) = h_1 \succ^{i_0} h_0$. □

Proof of Proposition 5

Let $(h_0, h_1, \ldots, h_k)$ be an improving path in H. We will show by induction that there is always at least one player $i \in N$ with $M_{h_k}(i) \neq M_{h_0}(i)$ and $h_k \succ^i h_0$. Thus, $h_k \neq h_0$.

$k = 1$: According to the definition of an improving path and because all constitutions satisfy protection against eviction, at least one of the deviating players strictly benefits from moving to h_1. Thus, there remains nothing to show.

$k > 1$: Suppose the statement is true for $k-1$. Note that $M_{h_{k-1}}(i) \neq M_{h_0}(i)$ and $h_{k-1} \succ^i h_0$ implies $M_{h_{k-1}}(i) \hat{\succ}^i M_{h_0}(i)$. Let $c_{k-1} \in M$ be the connection and $D_{k-1} \in \mathcal{D}^{c_{k-1}}_{h_{k-1}}$ be the subset of players with $h_k = h_{k-1} \pm (c_{k-1}, D_{k-1})$. First consider the case $i \in D_{k-1}$. By assumption every player $j \in D_{k-1}$ strictly benefits from the deviation. Because preferences are lexicographic, this implies not only $h_k \succ^i h_0$ but also $M_{h_k}(i) \neq M_{h_0}(i)$. Now suppose $i \notin D_{k-1}$. Then, of course, $M_{h_k}(i) = M_{h_{k-1}}(i) \neq M_{h_0}(i)$. But it might be possible that i suffers from this deviation, i.e., $h_{k-1} \succ^i h_k$. Nevertheless, because $M_{h_k}(i) = M_{h_{k-1}}(i) \hat{\succ}^i M_{h_0}(i)$ the player still strictly prefers h_k to h_0. □

Some of the following proofs use similar technical arguments and the following lemma will serve as a convenient and useful tool. Recall that for each $h \in \mathcal{H}$, $\mathcal{A}^c_h(\mathcal{C}) = \{D \in \mathcal{D}^c_h \mid \exists S \in \mathcal{S}^c_h(D) \text{ such that } h \pm (c, D) \succ^i h \ \forall i \in (D \setminus h(c)) \cup S\}$ is the set of all feasible deviations causing an instability in $c \in M$. We say that h is *exit-proof* if $D \in \mathcal{A}^c_h(\mathcal{C})$ implies $D \nsubseteq h(c)$ for all $c \in M$. Phrased differently, a network h is not exit-proof if and only if there exists a connection $c \in M$ and a group of members $D \subseteq h(c)$ which causes an instability.

Lemma 2. *Let $(N, M, \succeq, \mathcal{C})$ be a society. Moreover, let $h \in \mathcal{H}$ be an arbitrary network. Then there exists an exit-proof network $\bar{h} \in I(h)$.*

Proof. Let $c \in M$ such that there exists $D \in \mathcal{A}_h^c$ with $D \subseteq h(c)$. If such a connection does not exist, the network is exit-proof already and there remains nothing to show. Consider $h' := h - (c, D)$. If $h'(c)$ is not exit-proof, further subsets of players can be deleted from c until the connection is either empty or no coalition is supporting these deviations any more. This proceeding can be repeated for all connections and because N and M are finite, after finitely many steps an exit-proof network $\bar{h}$ will be reached. □

Note that by applying the previous result, Lemma 1 could be restated as follows: There exists no closed cycle if and only if, for each exit-proof network $\bar{h} \in \mathcal{H}$ that is not constitutionally stable, there exists an improving path leading from $\bar{h}$ to a constitutionally stable network.

Proof of Proposition 6

The main idea of the proof is to construct for every network in H an improving path leading from this network to a stable network. By closedness, this stable network is in H, too. Hence, there cannot be a closed cycle in H.

For constructing these paths, let us define, for each network $h \in H$, the set

$$\bar{M}_h = \{c \in M \mid \exists\, j \in h(c) : h \succ^j h - (c, \{j\})\}.$$

That is, a connection $c \in M$ is contained in $\bar{M}_h$ if and only if at least one of its members does not want to leave c. In particular, if $c \in \bar{M}_h$, due to equability, none of the members wants to leave c.

Let $h_1 \in H$ be an arbitrary network. By applying Lemma 2 we may assume that h_1 is exit-proof. In the following, we will establish that if h_1 is not constitutionally stable (if this would be the case, there would remain nothing to be shown), there exists an improving path from h_1 to another exit-proof network h_2 such that either $\bar{M}_{h_1} \subsetneq \bar{M}_{h_2}$ or $\bar{M}_{h_1} = \bar{M}_{h_2}$ and $h_1 \subsetneq h_2$. Then, if h_2 is not constitutionally stable, it is possible to iterate the previous step again and again. In particular, each time the step is iterated, either there are more connections whose members do not want to leave or the network strictly grows. Since both, the set of connections and the set of players, are supposed to be finite, this procedure will end after finitely many steps.

Case 1: There exists $c \in M \setminus \bar{M}_{h_1}$ with $\mathcal{A}_{h_1}^c \neq \emptyset$.

Note that because h_1 is exit-proof, $D \in \mathcal{A}_{h_1}^c$ if and only if $D \not\subseteq h_1(c)$, i.e., there is at least one player $i_1 \in D \setminus h_1(c)$ who joins the connection. Let $h_1' := h_1 \pm (c, D)$. Because all players are self-concerned, this implies:

$$h_1 \sim^{i_1} h_1 \pm (c, D \setminus \{i_1\}) = h_1' - (c, \{i_1\}).$$

In other words, after joining the connection, player i_1 has no incentive to leave it unilaterally. By equability this is also true for all $i \in h_1'(c)$ and, thus, $c \in \bar{M}_{h_1'}$. Now let $c' \in \bar{M}_{h_1}$. Note that $c \neq c'$ and $h_1(c') = h_1'(c')$. Therefore, equability implies that $c' \in \bar{M}_{h_1'}$, too. Moreover, assume there exists $D' \in \mathcal{A}_{h_1'}^{c'}$ with $D' \subseteq h_1'(c')$, that is, assume that c' is not exit-proof any more. Let $S' \in \mathcal{S}_{h_1'}^{c'}(D')$ be the corresponding supporting coalition. From regularity, it follows that there is a player $j \in S'$ with $h_1 \succeq^j h_1 - (c', D')$ but $h_1' - (c', D') \succeq^j h_1'$. If $j \notin D'$, this would contradict separability because $h_1(c') = h_1'(c')$. If $j \in D'$, this would violate equability and self-concern. Therefore, the assumption cannot be true or, in other words, transforming c does not affect the exit-proofness of c'. Similar considerations also apply if $c' \in M \setminus \bar{M}_{h_1}$ with $c' \neq c$. However, it might be possible that c itself is not exit-proof any more. In this case, we can delete (analogously to Lemma 2) all groups of players from the connection under the conditions that (i) no player joins c and (ii) all deviations comply with the constitutions, i.e., they are feasible and supported by a supporting coalition. Let h_2 be the network which is finally reached by means of this procedure. In particular, by advancing the same arguments as before it can be shown that the other connections are still exit-proof in h_2 and, moreover, $\bar{M}_{h_1} = \bar{M}_{h_2} \setminus \{c\} \subsetneq \bar{M}_{h_2}$.

Case 2: $\mathcal{A}_{h_1}^{c} = \emptyset$ for all $c \in M \setminus \bar{M}_{h_1}$.

Because h_1 is not constitutionally stable, there exists $c_1 \in \bar{M}_{h_1}$ with $\mathcal{A}_{h_1}^{c_1} \neq \emptyset$. Let $D \in \mathcal{A}_{h_1}^{c_1}$ be of minimal size, i.e., $\tilde{D} \notin \mathcal{A}_{h_1}^{c_1}$ for all $\tilde{D} \subsetneq D$. Moreover, let $S \in \mathcal{S}_{h_1}^{c_1}(D)$ be the corresponding coalition which supports the deviation of D. We will show first that $D \cap h_1(c_1) = \emptyset$, that is, there are only players in D who join the connection c. Assume this is not true, i.e., there exists $i \in D \cap h_1(c_1)$. Then, $h_1 \succ^i h_1 - (c_1, \{i\}) \sim^i h_1 \pm (c_1, D)$ by self–concern and the definition of $\bar{M}_{h_1}$. Thus, i would not support the deviation of D. Expressed differently, $S \cap D = \emptyset$. Because all constitutions are supposed to be decomposable and regular, we also have $\{i\} \in \mathcal{D}_{h_1}^{c_1}$ and $S \in \mathcal{S}_{h_1}^{c_1}(\{i\})$. By construction of h_1 the network is exit-proof and, therefore, there exists a player $j \in S$ with $h_1 \succeq^j h_1 - (c_1, \{i\})$. In particular, due to uniformity this is true for all members of S. But exploiting separability then yields $h_1 \pm (c_1, D \setminus \{i\}) \succeq^j h_1 \pm (c_1, D) \succ^j h_1$ for all $j \in S$ which contradicts the minimality of D.

Define $h_2 := h_1 + (c_1, D)$. Because all $i \in D$ agreed to joining c_1, $h_2 \succ^i h_1 \sim^i h_2 - (c_1, \{i\})$ by self-concern. Therefore, from equability it follows that no player in $h_2(c_1)$ wants to leave the connection unilaterally. Moreover, if

$\bar{D} \in \mathcal{D}_{h_2}^{c_1}$ with $\bar{D} \cap h_2(c_1) \neq \emptyset$, then

$$h_2 \succ^i h_2 - (c_1, \{i\}) \sim^i h_2 \pm (c_1, \bar{D}) \tag{4}$$

for all $i \in \bar{D} \cap h_2(c_1)$, again by self-concern. In other words, all players who would have to leave the connection would suffer from this deviation.
In the remainder of the proof we will show that h_2 is indeed exit-proof. Let $c' \in M$ be an arbitrary connection and $D' \in \mathcal{D}_{h_2}^{c'}$ with $D' \subseteq h_2(c')$. Recall that $\mathcal{D}_{h_2}^{c'} = \mathcal{D}_{h_1}^{c'}$ by regularity and, thus, $D' \in \mathcal{D}_{h_1}^{c'}$, too.

First consider the case $c_1 \neq c'$. Since the agents' preferences are separable, $h_2 \succeq^j h_2 - (c', D')$ if and only if $h_1 \succeq^j h_1 - (c', D')$ for all $j \in h_2(c') \setminus D'$. Therefore, if $j \in h_2(c') \setminus D'$ does not support the deviation of D' in h_1, the same holds for h_2, too. However, this is also true for all $j \in D'$ due to equability and self-concern. Hence, it follows that a coalition supports a deviation in h_2 if and only if it does the same in h_1 (cf. Case 1). In particular, this implies that the connection c' is also exit-proof in h_2.

Next consider $c' = c_1$. Here we have to distinguish two cases, $S = \emptyset$ and $S \neq \emptyset$. First consider $S = \emptyset$, that is, when deviating from h_1 to h_2, the agents in D do not need the consent of other members for entering c. Assume there exists $D' \in \mathcal{A}_{h_2}^{c_1}$ with $D' \subseteq h_2(c_1)$. Let $S' \in \mathcal{S}_{h_2}^{c_1}(D')$ be a coalition which supports the deviation of D', i.e., there is no $j \in S'$ with $h_2 \succeq^j h_2 - (c_1, D')$. From Equation (4) follows $D' \cap S' = \emptyset$. Moreover, regularity implies that there exists $\emptyset \neq S'' \in \mathcal{S}_{h_1}^{c_1}(D')$ with $S'' \subseteq S'$. Note that $h_2 - (c_1, D') = (h_1 + (c_1, D)) - (c_1, D') = h_1 \pm (c_1, D \pm D')$. In particular, $D' \subseteq h_1(c_1)$ if and only if $D \cap D' \neq \emptyset$. However, this is not possible because this would contradict separability of the players' preferences. Therefore, $D \cap D' \neq \emptyset$. But this is not possible, too: by decomposability and regularity also $D \cap D' \in \mathcal{D}_{h_2}^{c_1} \subseteq \mathcal{D}_{h_1}^{c_1}$ and $S' \in \mathcal{S}_{h_2}^{c_1}(D \cap D')$. Because $\emptyset \in \mathcal{S}_{h_1}^{c_1}(D \cap D')$, again decomposability and regularity implies $D \cap D' \subseteq S'$ which contradicts Equation (4). Next consider $S \neq \emptyset$. We will show that $|D| = 1$. Let $i \in D$. If there would be no player $j \in S$ with $h_1 + (c, \{i\}) \succ^i h_1$, decomposability together with separability would imply $h_1 + (c, D \setminus \{i\}) \succeq^j h_1 + (c, D) = h_2 \succ^j h_1$ for all $j \in S$. In other words, S would also support a deviation of $D \setminus \{i\}$. Moreover, from uniformity it follows $h_1 + (c, D \setminus \{i\}) \succeq^j h_1 + (c, D) = h_2 \succ^j h_1$ for all $j \in h_1(c) \cup (D \setminus \{i\})$. Thus, the players in $D \setminus \{i\}$ would agree to joining c without player i which would contradict minimality of D. Therefore, given that each $i \in D$ is supported by at least one player in S, from uniformity it follows that this is also true for all other members of $h_1(c_1)$. That is, $h_1 + (c_1, \{i\}) \succ^j h_1$ for all $j \in h_1(c_1)$ and, thus, $h_1 + (c_1, \{i\}) \succ^j h_1 \succ^j h_1 - (c_1, \{j\}) \sim^j (h_1 + (c_1, \{i\})) - (c_1, \{j\})$

because $c_1 \in \bar{M}_{h_1}$. By equability this also holds for player i or, phrased differently, i has an incentive for joining c_1 unilaterally. In fact, this implies $D = \{i\}$ by minimality of D. Moreover, by uniformity, all players in $h_1(c_1)$ strictly benefit from deviating from h_1 to h_2. Now let D', S', and S'' be given as in the case $S = \emptyset$. Then, as before we have $D' \cap D \neq \emptyset$ and, thus, $i \in D'$. By decomposability also $(h_1(c_1) \cap (D \pm D')) = h_1(c_1) \cap D' \in \mathcal{D}_{h_1}^{c_1}$ and $S'' \in \mathcal{S}_{h_1}^{c_1}(h_1(c_1) \cap D')$. Since we have $\bar{D} \in \mathcal{A}_{h_1}^{c_1}$ only if $\bar{D} \nsubseteq h_1(c_1)$, there exists $j \in S''$ with $h_1 \succeq^j h_1 - (c_1, h_1(c_1) \cap D')$. But this implies:

$$h_1 - (c_1, h_1(c_1) \cap D') = h_2 - (c_1, D') \succeq^j h_2 \succ^j h_1 \succeq^j h_1 - (c_1, h_1(c_1) \cap D')$$

which obviously is a contradiction. Thus, the assumption $D' \subseteq h_2(c_1)$, $D' \in \mathcal{A}_{h_1}^{c_1}$ must be false and c_1 is also exit-proof in h_2. □

Proof of Proposition 7

The proof proceeds in a similar way as the one of Proposition 6. As above we will construct for every exit-proof network $h_1 \in H$ an improving path leading to a stable network.

Step 1: In this step we establish that if h_1 is not constitutionally stable, there exists an improving path to another exit-proof network h_2 such that there is $D_1 \subseteq N$ with $h_2 \succ^i h_1$ and $M_{h_1}(i) \neq M_{h_2}(i)$ for all $i \in D_1$. Note that this implies $h_1 \neq h_2$. Therefore, suppose h_1 is not constitutionally stable. Then there exists $c_1 \in M$ with $\mathcal{A}_{h_1}^{c_1} \neq \emptyset$. Let $D_1 \in \mathcal{A}_{h_1}^{c_1}$ be of minimal size, i.e., $\tilde{D} \notin \mathcal{A}_{h_1}^{c_1}$ for all $\tilde{D} \subsetneq D_1$. Moreover, let $S \in \mathcal{S}_{h_1}^{c_1}(D_1)$ be the corresponding coalition which supports the deviation of D_1. We will show first that $|D_1| = 1$. Note that $D_1 \nsubseteq h_1(c_1)$ because h_1 is exit-proof by assumption. Moreover, for all $i \in D_1$ there is at least one $j \in S$ with $h_1 + (c_1, \{i\}) \succ^j h_1$. If this would not be satisfied, analogously to Case 2 in the proof of Proposition 6 we would have $D_1 \setminus \{i\} \in \mathcal{A}_{h_1}^{c_1}$ since the constitutions are decomposable and the preferences are separable and lexicographic. But this would contradict minimality of D_1. Therefore, given that each $i \in D_1$ is supported by at least one player in S, from uniformity it follows that this also holds for all other members of $h_1(c_1)$ and, thus, $D_1 = \{i\}$ by minimality of D_1. Moreover, by applying uniformity, all members in $h_1(c_1)$ are strictly better off if i enters the connection. Next we show that c_1 is also exit-proof in $\bar{h} := h_1 + (c_1, \{i\})$. Assume this is not true, that is, assume there exists $D' \in \mathcal{A}_{\bar{h}}^{c_1}$ with $D' \subseteq \bar{h}(c_1)$. Analogously to Case 2 in the proof of Proposition 6 we must have $i \in D'$ because the players's preferences are lexicographic and separable. Let $S' \in \mathcal{S}_{\bar{h}}^{c_1}(D')$ be a coalition which supports the deviation of D'. Moreover, let $S'' \in \mathcal{S}_{h_1}^{c_1}(D')$ with $S'' \subseteq S'$ be defined as in Case 2 in the proof of Proposition 6. Then, by advancing analog arguments as above we get

$$h_1 - (c_1, h_1(c_1) \cap D') = \bar{h} - (c_1, D') \succ^j \bar{h} \succ^j h_1 \succeq^j h_1 - (c_1, h_1(c_1) \cap D')$$

which obviously is a contradiction. Thus, the assumption $D' \subseteq \bar{h}(c_1)$, $D' \in \mathcal{A}_{h_1}^{c_1}$ must be false and c_1 is also exit-proof in $\bar{h}$.
Now, suppose there exists $c' \neq c_1$ with $\bar{D} \in \mathcal{A}_{\bar{h}}^{c'}$ for some $\bar{D} \subseteq \bar{h}(c') = h_1(c')$. Let $\bar{S} \in \mathcal{S}_{\bar{h}}^{c'}(\bar{D})$ be the corresponding supporting coalition. Note that $\bar{D} \cap \bar{S} \neq \emptyset$ because of separability. Let $i \in \bar{D} \setminus \bar{S}$. By decomposability and regularity also $\{i\} \in \mathcal{D}_{\bar{h}}^{c'} = \mathcal{D}_{h_1}^{c'}$ and $\bar{S} \in \mathcal{S}_{\bar{h}}^{c'}(\{i\})$. Since h_1 is exit-proof, there exists $j \in \bar{S}$ with $h_1 \succeq^j h_1 - (c', \{i\})$ and, thus, also $\bar{h} \succeq^j \bar{h} - (c', \{i\})$. Therefore, because the players' preferences satisfy uniformity, $\bar{h} \succeq^{\bar{j}} \bar{h} - (c', \{i\})$ for all $\bar{j} \in \bar{h}(c') \setminus \{i\}$. By exploiting separability this yields

$$\bar{h} - (c', \bar{D} \setminus \{i\}) \succeq^{\bar{j}} \left(\bar{h} - (c', \bar{D} \setminus \{i\})\right) - (c', \{i\}) = \bar{h} - (c', \bar{D}) \succ^{\bar{j}} \bar{h}$$

for all $\bar{j} \in \bar{S}$. Iterating this argument implies $\bar{D} \cap \bar{S} \in \mathcal{A}_{\bar{h}}^{c'}$, too, and $\bar{D} \setminus \bar{S} \notin \mathcal{A}_{\bar{h}}^{c'}$. Therefore, all players in $\bar{D} \cap \bar{S} \in \mathcal{A}_{\bar{h}}^{c'}$ strictly benefit from this deviation. Note that it might be the case that there exists $j \in \bar{h}(c') \cap D$ who is worse off after this change of the connection. However, because the preferences are lexicographic, this player still strictly prefers $\bar{h} - (c, \bar{D} \cap \bar{S})$ to h_1. By iterating these arguments all subsets of members where all players agree to deviate can be deleted from all connections. Let h_2 be the network which is finally reached by means of this procedure. In particular, because of separability and uniformity, h_2 is exit-proof, too. Moreover, since no player has to leave a connection against her will and preferences are lexicographic, all players who deviated strictly prefer h_2 to h_1.

Step 2: In this step we show that if h_2 is not stable, there exists

(i) a sequence of non-empty subsets $D_1, D_2, \ldots, D_{k-1}$, and

(ii) a sequence of exit-proof networks $h_1, h_2, h_3, \ldots, h_k$ such that there is an improving path from h_{l-1} to h_l for all $2 \leq l \leq k$ and the following two conditions are satisfied:

   (a) $h_l \succ^i h_{l'}$ for all $2 \leq l \leq k$, $1 \leq l' \leq l-1$, and $i \in D_{l-1}$;

   (b) if $h_l \not\succ^i h_{l-1}$, then $M_{h_l}(i) = M_{h_{l-1}}(i)$.

In particular, (a) implies $h_k \neq h_{l'}$ for all $1 \leq l' < k$. Therefore, since there are only finitely many exit-proof networks, this sequence will stop after finitely many steps and, thus, the last one has to be stable.

We will show the existence of the sequence by means of induction. For $k = 2$ see Step 1. Consequently, let $k \geq 3$ and assume there exist $h_3, \ldots, h_k$ and $D_2, \ldots, D_{k-1}$ as defined above. Moreover, suppose h_k is not stable. Since this network is exit-proof by assumption, there exists $c_k \in M$ with $\mathcal{A}^{c_k}_{h_k} \neq \emptyset$ and $D \nsubseteq h_k(c_k)$ for all $D \in \mathcal{A}^{c_k}_{h_k}$. Let $D_k \in \mathcal{A}^{c_k}_{h_k}$ be of minimal size and construct h_{k+1} analogously to h_2 in Step 1. Similar to above, players deviate only if they have a strict incentive and $h_{k+1} \succ^i h_k$ for all $i \in D_k$. First, this implies $M_{h_k}(i) = M_{h_{k+1}}(i)$ for all $i \in N$ with $h_{k+1} \not\succ^i h_k$. Second, if $i \in D_k \cap D_{k-i}$, then clearly $h_{k+1} \succ^i h_{l'}$ for all $1 \leq l' \leq k$ because of transitivity. Therefore, let $i \in D_{k+1} \setminus D_k$. If $M_{h_k}(i) = M_{h_{l'}}(i)$ for all $1 \leq l' \leq k$, we have $h_{k+1} \succ^i h_{l'}$ for each of these networks because i's preferences are lexicographic. On the other hand, if $M_{h_k}(i) \neq M_{h_1}(i)$, let $l_1 := \min\left\{2 \leq l \leq k \mid M_{h_{l-1}}(i) \neq M_{h_l}(i)\right\}$. Note that (ii) implies $h_{l_1} \succ^i h_{l_1-1}$. Thus, from this also follows $h_{l_1} \succ^i h_{l'}$ for all $1 \leq l' \leq l_1 - 1$ by lexicography. Next consider $l_2 := \min\left\{l_1 + 1 \leq l \leq k \mid M_{h_{l-1}}(i) \neq M_{h_l}(i)\right\}$. By advancing analog arguments as before we get $h_{l_2} \succ^i h_{l'}$ for all $1 \leq l' \leq l_2 - 1$ and, thus, iterating the procedure yields $h_{k+1} \succ^i h_{l'}$ for all $1 \leq l' \leq k$. □

Proof of Proposition 8

This proof proceeds similarly as the proofs of the two previous propositions. Again, we construct for every network which is in $H = \{h \in \mathcal{H} \mid O \cap h(c) = \{o_c\}\ \forall\, c \in M\}$ an improving path leading from this network to a stable network. Because H is closed, this stable network has to be in H, too. Therefore, let $h_1 \in H$ be an arbitrary network. Because of Lemma 2 we may assume that h_1 is exit-proof. Moreover, let $c_1 \in M$ be an arbitrary connection with $\mathcal{A}^{c_1}_{h_1} \neq \emptyset$. The construction of the path proceeds in three steps.

Step 1: We establish that there exists $B_1 \in \mathcal{A}^{c_1}_{h_1}$ with $\mathcal{A}^{c_1}_{h_1+(c_1,B_1)} = \emptyset$.

The main idea of this step is to exploit separability of the owner's preferences. Let

$$B_1 := \left\{i \in E \setminus h_1(c_1) \;\middle|\; h_1 + (c_1, \{i\}) \succ^i h_1 \text{ and } h_1 + (c_1, \{i\}) \succ^{o_{c_1}} h_1 \right\}.$$

That is, B_1 contains exactly those players who want to join c_1 and would be accepted by o_{c_1}. Let $i, j \in B_1$. Then, $h_1 + (c_1, \{i, j\}) \succ^{o_{c_1}} h_1 + (c_1, \{i\}) \succ^{o_{c_1}} h_1$ by separability of o_{c_1}'s preferences. Iterating this argument implies $h_1 + (c_1, B_1) \succ^{o_{c_1}} h_1$. Moreover, since the workers' preferences are lexicographic, also $h_1 + (c_1, B_1) \succ^i h_1$ for all $i \in B_1$. Thus, $B_1 \in \mathcal{A}^{c_1}_{h_1}$. Now suppose there exists $D \in \mathcal{A}^{c_1}_{h_1+(c_1,B_1)}$. If $D \subseteq h_1(c_1) \cup B_1$, the definition of B_1 and exit-proofness of h_1 imply $h_1 + (c_1, B_1) \succeq^{o_{c_1}} (h_1 + (c_1, B_1)) - (c_1, i)$ for all $i \in$

$h_1(c_1) \cup B_1$. Advancing the same arguments as before yields $h_1 + (c_1, B_1) \succeq^{o_{c_1}} (h_1 + (c_1, B_1)) - (c_1, D)$, which implies that o_{c_1} would not support the deviation. Moreover, the workers in $h_1(c_1) \cup B_1$ obviously do not want to leave the firm and thus, $D \subseteq h_1(c_1) \cup B_1$ cannot be true. However, if $D \nsubseteq h_1(c_1) \cup B_1$ and there exists $i \in D \setminus h_1(c_1)$ with $(h_1 + (c_1, B_1)) + (c_1, \{i\}) \succ^{o_{c_1}} h_1 + (c_1, B_1)$, then by construction of B_1 and because i's preferences are lexicographic, this worker would not agree to join c_1. Therefore, $\mathcal{A}^{c_1}_{h_1+(c_1,B_1)}$ must be empty.

Step 2: We construct an improving path leading from $h'_1 := h_1 + (c_1, B_1)$ to another exit-proof network h_2 with $h_2 \succ^i h_1$ for all $i \in B_1$ and $h_2 \succeq^i h_1$ for all $i \in E \setminus B_1$. Let $c' \in M$ such that there exists $B' \subseteq h'_1(c')$ with $B' \in \mathcal{A}^{c'}_{h'_1}$ and choose B' maximal with respect to "$\subseteq$", i.e., there exists no $\bar{B} \subseteq h'_1(c')$ with $\bar{B} \in \mathcal{A}^{c'}_{h'_1}$ and $B' \subsetneq \bar{B}$. Note that $c' \neq c_1$ because $\mathcal{A}^{c_1}_{h'_1} = \emptyset$. By assumption $o_{c'}$'s preferences are separable and, thus, $h'_1 \succeq^{o_{c'}} h'_1 - (c', B')$ by exit-proofness of h_1. Therefore, $h'_1 - (c', B') \succ^j h'_1$ for all $j \in B'$. Now suppose there exists $i \in B' \setminus B_1$. This implies $i \in h_1(c)$ if and only if $i \in h'_1(c)$ for all $c \in M$. If i has a strict incentive for leaving c' in h'_1, she would also have a strict incentive for leaving the connection in h_1 because her preferences are lexicographic. But this contradicts exit-proofness of h_1 and, thus, $B' \subseteq B_1$. Moreover, by construction of B' and separability of $o_{c'}$'s preferences, there exists no further set of workers $B'' \subseteq h'_1(c') \setminus B'$ with $B'' \in \mathcal{A}^{c'}_{h'_1-(c',B')}$. By iterating the previous procedure, it is possible to reach an exit-proof network h_2 by deleting all workers from all connections they want to leave without impairing the other workers in $E \setminus B_1$. In particular, for all $i \in E \setminus B_1$ nothing changes and, therefore, they are indifferent between h_2 and h_1. However, all $i \in B_1$ strictly benefit from the deviations and thus, they strictly prefer h_2 to h_1.

Step 3: Iterating the procedure.

Once at h_2, if $A^c_{h_2} = \emptyset$ for all $c \in M$, there remains nothing to show. Therefore, assume there exists $c_2 \in M$ with $A^{c_2}_{h_2} \neq \emptyset$. By repeating Steps 1 and 2 it is possible to find $B_2 \subseteq E \setminus h_2(c_2)$ with $\mathcal{A}^{c_2}_{h_2+(c_2,B_2)} = \emptyset$ and to construct an improving path leading from $h_2 + (c_2, B_2)$ to an exit-proof network h_3. Analogously, h_2 will be Pareto dominated by h_3 from the workers' perspective. Because H is finite, there exists only a finite number of exit-proof networks. Hence, this procedure will end after finitely many steps. □

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