Stochastic Costly Signaling with Exogenous Information

Tom Truyts*

Abstract

How does better exogenous information affect costly signaling? A game of stochastic costly signaling in the presence of exogenous imperfect information is shown to have a unique equilibrium. More accurate exogenous information either decreases or increases equilibrium signaling, depending on whether prior beliefs are respectively above or below a unique threshold level. More accurate exogenous information can induce a less informative signaling equilibrium, and can result in a lower expected accuracy of the uninformed party's equilibrium beliefs.

Keywords: Asymmetric Information, Monotonic Costly Signaling, Stochastic Signaling, Noisy Signaling, Advertising, Job Market Signaling, Conspicuous Consumption

JEL: C72, D82

1 Introduction

Costly signaling models explain ostentatious waste as a way of communicating private information that otherwise cannot be credibly communi-

*Saint-Louis University - Brussels, CEREC, Boulevard du Jardin Botanique 43, 1000 Bruxelles, Belgium; University of Leuven, CES, Naamsestraat 69, 3000, Leuven, Belgium; Université Catholique de Louvain, CORE, Voie du Roman Pays 34, 1348 Louvain-la-Neuve, Belgium. E-mail tom.truyts@fusl.ac.be, tel. +3222117845.
I thank Erik Schokkaert, Frans Spinnewyn, Johannes Spinnewijn, Koen Decancq, Vincent Vannetelbosch and seminar and conference participants in Brussels, Louvain-la-Neuve, Bloomington, Belfast, Namur, Willemstad, Paris, Leuven and Monte Verità. Tom Truyts is postdoctoral research fellow of the F.R.S.-FNRS. Part of this research was funded by the Flemish Science Foundation (project G.0522.09N).

cated, and have found numerous applications in recent decades.[1] Policy makers and economists have since long deplored the welfare losses due to conspicuous waste, and occasionally applauded the welfare gains associated with the resulting information transfer (e.g. by solving Akerlof's (1970) market for lemons problem).[2] But what happens to ostentatious waste if better information is exogenously provided to the uninformed parties (the 'receiver')? A common (but false) intuition is that better information about the subject of the informed party's (the 'sender') private information is generally an efficient way of reducing wasteful signaling.

Veblen (1899(1994), pp. 53-55) observed that "*Conspicuous consumption claims a relatively larger portion of the income of the urban than of the rural population, and the claim is also more imperative.* [...] *So it comes, for instance, that the American farmer and his wife and daughters are notoriously less modish in their dress, as well as less urbane in their manners, than the city artisan's family with equal income.* [...] *And in the struggle to outdo one another the city population push their normal standard of conspicuous consumption to a higher point* [...]." Veblen suggested the availability of exogenous information as an explanation. "*The means of communication and the mobility of the population now expose the individual to the observation of many persons who have no other means of judging of his reputability than the display of goods [...]. One's neighbors, mechanically speaking, often are socially not one's neighbors, or even acquaintances; and still their transient good opinion has a high degree of utility.*" If the exogenous information is perfect, Veblen's intuition is trivially true: if exogenous information resolves the information asymmetry, one expects no costly signaling. But how does equilibrium signaling depend on the accuracy of exogenous information when both signaling and the exogenous information are imperfect? And what happens to the expected accuracy of receiver's equilibrium

[1] See e.g. Riley (2001) for a survey of the economic literature. Examples include labor economics (Spence, 1973), advertising (Milgrom and Roberts, 1986), finance (Myers and Majluf 1984, John and Williams, 1985, Bhattacharya 1979), animal behavior and morphology (Zahavi, 1975, Grafen, 1990a,b) and consumption (Frank, 1999; or Truyts (2010) for a recent survey).

[2] See Truyts (2012) and the references therein for a discussion of various policies proposed for reducing the welfare costs of signaling.

beliefs, if exogenous information becomes more accurate?

Real world costly signals are usually imperfect information sources, and the receiver usually has other information (beyond the sender's control) about the subject of asymmetric information. In a job market example, an academic degree can imperfectly reflect a job candidate's productivity because of luck with examination questions, a bad day during the exams or an employers' hardship to judge a program's difficulty. Moreover, employers often observe additional information: they often use psychometric tests during recruitment or learn about the candidate from social relations. An important distinction is whether the sender knows the actual realization of exogenous imperfect information when choosing a signaling strategy. If she does (e.g. ethnic markers in job market signaling), more accurate exogenous information alters the prior beliefs in equilibrium. Such marginal changes in prior beliefs were studied in e.g. Matthews and Mirman (1983) and Jeitschko and Normann (2012). This article's focus is on cases in which the sender knows the accuracy of exogenous information, but not its realization (e.g. psychometric tests during recruitment).

This article develops a stochastic signaling model with exogenous imperfect information, and thus relates to a small literature on stochastic costly signaling. Matthews and Mirman (1983) introduced noise in terms of demand shocks in a limit pricing model and demonstrate a number of advantages of stochastic signaling games: a limited number of equilibria, smooth comparative statics and a solution that depends on prior beliefs.[3] Carlsson and Dasgupta (1997) develop vanishing noise as an equilibrium selection criterion for non-stochastic signaling games. De Haan et al. (2011) and Jeitschko and Normann (2012) test the implications of stochastic signaling models experimentally.

In what follows, a sender has binary private information and chooses a signal from the real line. The receiver observes this signal distorted

[3] Note that these three points are major problems of non-stochastic signaling models (e.g. Spence, 1973, Riley, 1979). See Mailath et al. (1993) for a critique of this last feature of non-stochastic costly signaling games.

by random noise, and also sees a binary exogenous imperfect signal. After observing both pieces of information, the receiver chooses an action from a continuum. Next to exogenous information, this constitutes a second key difference to the models of Matthews and Mirman (1983), de Haan et al. (2011) and Jeitschko and Normann (2012), who assume that the receiver makes a binary choice. Under some mild regularity conditions, the existence and uniqueness of a sequential equilibrium is established for this setting by means of an elementary application of the Poincaré-Hopf index theorem. In equilibrium, both sender types engage in signaling: the high type tries to distinguish herself more, while the low type tries to restore the receiver's confusion. The effect of more accurate exogenous information on the unique signaling equilibrium is characterized in two ways. First, equilibrium signaling is shown to be non-monotonic with respect to the accuracy of exogenous information. A threshold level of prior beliefs separates the cases where costly signaling increases or decreases with more accurate exogenous information. An interval of sufficiently low prior beliefs generically exists at which more accurate exogenous information makes both sender types signal more. Second, more accurate exogenous information can result in a lower expected accuracy of receiver's equilibrium beliefs, due to the changes it induces in equilibrium signaling.

The main difference to earlier work on costly signaling in the presence of exogenous information lies in the imperfect observation of signals. This implies a smooth dependence of equilibrium signaling on the accuracy of exogenous information, rather than a discrete shift from separating to (semi-) pooling equilibria. For non-stochastic signaling with exogenous information, Feltovitch et al. (2002) show the existence of a non-monotonic signaling equilibrium: middle types signal while the high and low types pool at zero signaling, if high types can sufficiently rely on exogenous information to separate them from the low types. Daley and Green (2012) show that separating equilibria do not survive the common stability-based equilibrium refinements (e.g. D1) in the presence of sufficiently informative exogenous imperfect information. Frank (1985) studies status consumption as an imperfect signal of ability in the presence of exogenous imperfect information, and concludes that if uninformed parties aggregate both information sources linearly by means of a minimum variance unbiased estimator, "*the ability-signaling rationale* [...] *suggests that incentives to distort consumption in favor of observable goods will be inversely related to the amount and reliability of independent information that exists concerning individual abilities*".

In the context of costless signaling (as in Crawford and Sobel (1982)), Chen (2012) shows the receiver's payoff is non-monotonic in the accuracy of exogenous information (i.e. it decreases discretely where the sender shifts from full revelation to babbling), while Blume et al. (2007) and Blume and Board (2009) introduce noise on the sender's message and show that such noise can be welfare enhancing.

Note that all models with exogenous information quoted above, as well as Veblen's observations, suggest that more accurate exogenous information enables the receiver to distinguish more between different sender types, and thus reduces the sender's (need for) investments in costly signaling.

This paper is structured as follows. The second section introduces the formal setting and suggests some specific examples. The third section characterizes equilibrium signaling in the presence of exogenous imperfect information. The final section concludes. All proofs are collected in a mathematical appendix.

2 Setting

A player, the sender, has private information about a quality parameter θ ('her type'), which is either high θ^H or low θ^L. She cares about the beliefs of an uninformed player, the receiver, about θ. The receiver has prior belief $p \in (0,1)$ that θ is high, and deems θ low with probability $1-p$. The sender sends a costly signal $s \in \bar{\mathbb{R}}_+$. As in Carlsson and Dasgupta (1997), the receiver observes this signal imperfectly as y, the sum of s and random noise ε:

$$y = s + \varepsilon. \tag{1}$$

Noise term ε is independently distributed according to a density function φ, with $E(\varepsilon) = 0$ and a variance which is finite and bounded away from zero. Assume that φ satisfies the following properties.

Condition 1 *Let φ be a C^2 probability density function which*

1. (symmetry) *is symmetric around the mean,*
2. (strict monotone likelihood ratio property - MLR) *is such that the ratio $\frac{\varphi(\varepsilon|\mu)}{\varphi(\varepsilon|\mu')}$ strictly increases with ε everywhere for two means $\mu > \mu'$,*[4]
3. (support) *has full support on $\mathbb{R}$.*

Prominent examples of distributions satisfying condition 1 are the normal and logistic distributions. Continuous differentiability, full support and MLR are in line with Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011). Full support on $\mathbb{R}$ implies that all y have an equilibrium interpretation, such that specifying out-of-equilibrium beliefs and the resulting multitude of equilibria is no cause of concern for this model.

The receiver observes two pieces of imperfect information about θ: distorted signal y and exogenous imperfect information ω, the distribution of which is independent of the sender's signaling. Assume for simplicity binary exogenous information

$$\omega \in \{L, H\},$$

of which the accuracy is denoted $q \in \left(\frac{1}{2}, 1\right)$, such that $q \equiv \Pr\left(\omega = H|\theta^H\right) = \Pr\left(\omega = L|\theta^L\right)$.

The sender's preferences are represented by a utility function

$$u(s, y, \omega|\theta, \beta) = v(s|\theta) + \kappa\beta(y, \omega) \tag{2}$$

[4]Note that this is equivalent to log-supermodularity of φ w.r.t. ε and μ, i.e. that for $\varepsilon > \varepsilon'$ and $\mu > \mu'$: $\varphi(\varepsilon|\mu)\varphi(\varepsilon'|\mu') > \varphi(\varepsilon'|\mu)\varphi(\varepsilon|\mu')$. See a.o. Karlin and Rubin (1956) or Athey (2002).

in which $\beta(y, \omega)$ represents the receiver's posterior 'believed' probability of the sender being a high type (her 'beliefs'), given the pair of imperfect signals (y, ω). Parameter $\kappa > 0$ represents the sender's constant marginal utility of β. I restrict the utility function as follows.

Condition 2 *Let v be C^2 with $v_1(0|.) > 0$, $v_{12}(.) > 0$ and $v_{11}(.) < \eta$ for an $\eta < 0$.*

Condition 2 imposes a standard Spence-Mirrlees single crossing condition, and ensures that both sender types have a unique utility maximizing choice of s in the absence of signaling concerns, denoted $\bar{s}^H$ and $\bar{s}^L$, such that $\bar{s}^H > \bar{s}^L > 0$.[5] The utility function in (2) departs from the utility function in Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and Jeitschko and Normann (2012) in two respects. First, it takes the receiver's beliefs directly as an argument. This either represents a problem in which the sender cares about the receiver's beliefs directly, or is shorthand notation by omitting an explicit analysis of the receiver's optimal choice of action in function of her beliefs. The receiver's choice is easily introduced explicitly, as illustrated at the end of this section. Second, (2) assumes that the sender's utility is strictly increasing with β, which reflects that the receiver's choice set is a continuum.[6] The fact that (2) is linear in β and additively separable in β and v may seem restrictive at first sight. But other than ensuring tractability, this formulation also aims to focus on the interaction between imperfect signaling and imperfect exogenous information by maximally separating the uncertainty associated with noisy information transmission from attitudes towards risk and other particularities in the utility functions of the sender and receiver.

[5] The assumption that the myopic choices of both sender types differ and are unique follows e.g. Mailath (1987), Matthews and Mirman (1983) and Jeitschko and Normann (2012), and is not crucial for the results and intuitions developped below. Indeed, a pure costly signaling game with linear signaling costs, in which both sender types choose $s^L = s^H = 0$ in the myopic optimum, is analytically more involved, but produces similar results.

[6] In the stochastic signaling models listed above, the receiver has a binary choice, which results in combination with MLR in a cut-off strategy as best reply: the receiver chooses the action most preferred by the sender if $y \geq y^*$, with y^* an optimally chosen threshold.

The sender maximizes expected utility, considering all possible realizations of ε and ω for given beliefs β:

$$\mathrm{E}u\left(s,y,\omega|\theta,\beta\right)=v\left(s|\theta\right)+\kappa\bar{B}\left(s|\theta,\beta\right), \tag{3}$$

with

$$\bar{B}\left(s|\theta,\beta\right)\equiv\sum_{\omega'\in\{L,H\}}\int\Pr\left(\omega=\omega'|\theta\right)\beta\left(y,\omega'\right)\varphi(y|s)dy.$$

We consider pure strategy sequential equilibria (S.E.) of the stochastic signaling game.[7] Let s^L and s^H denote respectively the (pure) signaling strategy of the low and high sender type. The receiver's beliefs are consistent with pure strategy profile $\left(s^L,s^H\right)$ if they satisfy Bayes' rule for each (y,ω):

$$\begin{aligned}\beta\left(y,H\right)&=\frac{pq\varphi\left(y|s^H\right)}{\left(1-q\right)\left(1-p\right)\varphi\left(y|s^L\right)+pq\varphi\left(y|s^H\right)}\\&=\left(1+\frac{1-q}{q}\frac{1-p}{p}\frac{\varphi(y|s^L)}{\varphi\left(y|s^H\right)}\right)^{-1} \qquad (4)\\ \beta\left(y,L\right)&=\frac{p\left(1-q\right)\varphi\left(y|s^H\right)}{\left(1-p\right)q\varphi\left(y|s^L\right)+pq\varphi\left(y|s^H\right)}\\&=\left(1+\frac{q}{1-q}\frac{1-p}{p}\frac{\varphi(y|s^L)}{\varphi(y|s^H)}\right)^{-1} \qquad (5)\end{aligned}$$

Note in (4) and (5) that MLR imposes consistent posterior beliefs $\beta\left(y,\omega\right)$ to be strictly monotonic with y if $s^L\neq s^H$.

If $q=\frac{1}{2}$ and $Var\left(\varepsilon\right)=0$, this game reduces to a standard costly signaling game with quasilinear preferences and separation in the myopic optimum. A number of textbook examples in the literature are easily adapted to this setting of stochastic signaling with imperfect exogenous

[7] A Sequential Equilibrium (S.E.) is described by a pair of strategy profile and posterior beliefs $\left(\left(\hat{s}^L\left(q\right),\hat{s}^H\left(q\right)\right),\beta\right)$, such that:
1. $\left(\hat{s}^L\left(q\right),\hat{s}^H\left(q\right)\right)$ maximizes expected utility (3) of each type given β
2. Beliefs $\beta\left(y,\omega\right)$ are Bayesian consistent with equilibrium strategies $\left(\hat{s}^L\left(q\right),\hat{s}^H\left(q\right)\right)$ as in (4) and (5).

information.

Example 1 (Status Signaling) *The sender wishes to signal her income θ to other consumers because she cares directly about their beliefs and esteem. The sender divides her income between invisible rest consumption and visible status consumption s, such that her utility is represented by $v^{SS}(\theta - s, s) + \kappa\beta(y, \omega)$. The 'intrinsic' utility of consumption, v^{SS}, is strictly increasing in both arguments and strictly concave. Status consumption is an imperfect signal because status goods can be bought at a discount price, second hand or can be cheap imitations, and because there are far too many visible consumption goods to keep track of prices. On the other hand, one can typically rely on gossip for additional information ω about a consumer's reputability.*

Example 2 (Job Market Signaling) *As in Spence (1973), the sender is a job candidate of high or low productivity θ. She invests in education s at cost $-(s-\theta)^2$. Hence, job candidates intrinsically enjoy some education up to θ for its own sake.*[8] *The receiver is an employer in a competitive job market, who sees a noisy educational score y and an additional imperfect test result ω and offers in equilibrium a contract with wage $\theta^L + \beta(y, \omega)(\theta^H - \theta^L)$. The expected utility of a job candidate is then $\theta^L - (s-\theta)^2 + (\theta^H - \theta^L)\bar{B}(s|\theta, \beta)$. Education is an imperfect signal because the sender may have been lucky with exam questions or have had a bad day during the exams, or an employer may have difficulty judging the difficulty of a degree. On the other hand, the employer typically has extra psychometric tests at her disposal during the recruitment stage, or can ask social relations whether they know more about the job candidate. Note that the wage does generically not equal the true productivity of the sender in this stochastic job market signaling game. The return to education thus only concerns a period needed by employers to learn about the sender's true productivity and to alter a possibly rigid contract.*

Example 3 (Advertising) *As in Milgrom and Roberts (1986) and Hertzendorf (1993), the sender is a monopolist, selling a new product of high or*

[8] As stressed above, this assumption is not crucial for our main results, but simplifies the analysis considerably.

low quality θ to a continuum of consumers, distributed uniformly on $[0,1]$ *. For simplicity, we take the commodity price as exogenously fixed. Before launching the new product, the sender can invest in advertising s at strictly convex costs $-v^{AD}(s|\theta)$ in a first period.*[9] *Advertising is an imperfect signal because consumers typically fail to observe the total number of advertisements bought, ignore their costs and have difficulty comparing the importance of these advertising costs to the size of the firm and market. They can often also rely on product tests in magazines or discussions on the internet. After observing both imperfect signals* (y,ω), *consumers decide whether or not to buy the product. Consumers buy the product if they deem the probability of a high quality product higher than their position on* $[0,1]$.[10] *Only consumers who buy the product observe the true quality θ, and can buy the product again in a second period (they all do if θ is high). If each consumer draws an independent y and ω, and profits per unit sold are π_θ (with $2\pi_H > \pi_L$), then profits of a high and low quality monopolist are respectively $-v^{AD}\left(s|\theta^H\right) + 2\pi_H \bar{B}\left(s|\theta^H,\beta\right)$ and $-v^{AD}\left(s|\theta^L\right) + \pi_L \bar{B}\left(s|\theta^L,\beta\right)$, such that the sender's preferences are represented by $\bar{B}\left(s|\theta^H,\beta\right) - \frac{v^{AD}\left(s|\theta^H\right)}{2\pi_H}$ and $\bar{B}\left(s|\theta^L,\beta\right) - \frac{v^{AD}\left(s|\theta^L\right)}{\pi_L}$.*

3 Equilibrium analysis

Before presenting the main results, this section first highlights a few simple features of the stochastic signaling game under consideration. First, by condition 1, the receiver's consistent beliefs are never degenerate for finite y and s, such that the receiver's best choice is generically suboptimal with respect to the sender's true type. In expectation, the weighted average of the receiver's consistent beliefs equals the prior belief, as stated by the following lemma.

[9] Note that by condition 2, $v_1^{AD}(0|.) > 0$. This can reflect other advantages of advertising (informing consumers of the existence of the product, entry deterrence...) as summarized in Bagwell (2007).

[10] If a risk neutral consumer's willingness to pay for a high and low quality product is resp. $\lambda^H > \lambda^L > 0$, she buys at price γ if $\beta(y,\omega)\, 2\left(\lambda^H - \gamma\right) + (1-\beta(y))\left(\lambda^L - \gamma\right) \geq 0$, i.e. if $\beta(y,\omega) \geq \frac{\gamma-\lambda^L}{\left(\lambda^H-\gamma\right)+\left(\lambda^H-\lambda^L\right)} \equiv \zeta$. We thus assume ζ uniformly distributed on $[0,1]$. Consumers with ζ negative or greater than 1 never and always buy, respectively. See also Milgrom and Roberts (1986) and Hertzendorf (1993).

Lemma 1 *If the receiver's beliefs* β *are consistent with strategies* $\left(s^L, s^H\right)$,

1. *the stochastic signaling game is zero sum in* B*:*

$$p\bar{B}\left(s^H|\theta^H, \beta\right) + (1-p)\,\bar{B}\left(s^L|\theta^L, \beta\right) = p, \tag{6}$$

2. $\bar{B}$ *depends only on* $\Delta \equiv s^H - s^L$ *and not on actual levels of* s.

Using lemma 1, it will be convenient to define the receiver's expected beliefs about a high sender type in terms of the difference in signals Δ and for beliefs consistent with a strategy profile $(0, \Delta)$, such that

$$B\left(\Delta|p,q\right) \equiv \int \left(q\tilde{\beta}\left(y,H|p,q\right) + (1-q)\,\tilde{\beta}\left(y,L|p,q\right)\right)\varphi(y|\Delta)dy, \tag{7}$$

with

$$\tilde{\beta}\left(y,H|p,q\right) = \left(1 + \frac{1-q}{q}\frac{1-p}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1} \tag{8}$$

and

$$\tilde{\beta}\left(y,L|p,q\right) = \left(1 + \frac{q}{1-q}\frac{1-p}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1}. \tag{9}$$

Note then that $\bar{B}\left(\Delta|\theta^H, \tilde{\beta}\right) = B\left(\Delta|p,q\right)$, while by (6)

$$\bar{B}\left(0|\theta^L, \tilde{\beta}\right) = \frac{p}{1-p}\left(1 - B\left(\Delta|p,q\right)\right). \tag{10}$$

Let $B'\left(\Delta|p,q\right)$ denote the marginal effect of Δ on B whilst keeping beliefs fixed (i.e. consistent with $(0,\Delta)$, such that

$$B'\left(\Delta|p,q\right) \equiv \int \left(q\tilde{\beta}\left(y,H|p,q\right) + (1-q)\,\tilde{\beta}\left(y,L|p,q\right)\right)\varphi_2\left(y|\Delta\right)dy,$$

and let $B_1\left(\Delta|p,q\right)$ denote the usual first order derivative to Δ. Unless potentially confusing, the two last arguments p and q are omitted from B, B' and $\tilde{\beta}$ to economize on notation. The next lemma shows that B increases with Δ.

Lemma 2 *If* φ *satisfies condition 1, then* $\Delta > 0$ *implies* $B'\left(\Delta\right) > 0$ *and* $B_1\left(\Delta\right) > 0$*, while* $B'\left(0\right) = 0$.

As such, the stochastic signaling game can be understood as an arms race in which both sender types waste means to secure for themselves a larger share of a given resource of fixed size: the receiver's expected consistent beliefs. The division of this resource depends only on the difference in signaling efforts Δ. The high sender type can create more distinction in expectation by increasing s^H, while the low sender type can increase signaling s^L to confuse the receiver more and undo the expected distinction established by the high type. If both sender types signal, then an amount $\min\left\{s^L - \bar{s}^L, s^H - \bar{s}^H\right\}$ is wasted, in the sense that exogenously reducing the signaling efforts of both sender types by this much (and adapting beliefs accordingly) improves the welfare of both sender types without affecting the information transferred to the receiver in expectation. Note also that $B(\Delta)$, the expectation of the receiver's believed probability that the high sender type is a high type, measures the expected accuracy of receiver's consistent beliefs. If the receiver's choice of action is strictly monotonic in β, then a higher B brings the receiver's choice on average closer to her optimal choice under full information about θ.

Equilibrium signaling strategies $\left(\hat{s}^L(q), \hat{s}^H(q)\right)$ are maximizing the sender's expected utility given the receiver's beliefs, which are in turn consistent with these strategies. Substituting (7) and (10) in the sender's problem (3) and differentiating to s (while taking beliefs as given), one obtains two standard first order conditions, which equate the marginal costs and benefits of signaling:

$$v_1\left(\hat{s}^H(q)\,|\theta^H\right) + \kappa B'\left(\hat{\Delta}(q)\right) = 0 \tag{11}$$

and

$$v_1\left(\hat{s}^L(q)\,|\theta^L\right) + \kappa\frac{p}{1-p}B'\left(\hat{\Delta}(q)\right) = 0. \tag{12}$$

Equations (11) and (12) characterize an equilibrium if the sender's problem is strictly concave for all strategy profiles. An extensive characterization of concavity in terms of the model's primitives is provided in appendix A.3, but in essence this condition requires that v is sufficiently concave and κ is not too large, as B can be either concave or convex. The following proposition shows that if this condition is satisfied, a unique S.E. exists. In this equilibrium, the high sender type signals strictly more than the low type. As in Jeitschko and Normann (2012) and unlike in non-stochastic costly signaling games in line with Spence (1973), signaling causes distortion at the top and bottom. The low sender type wastes means in equilibrium to confuse the receiver and undo distinction with the high sender type.

Proposition 1 *If φ and v satisfy respectively conditions 1 and 2 and if the sender's problem is strictly concave, then a unique S.E. in pure strategies exists, in which equilibrium strategies $\left(\hat{s}^H(q), \hat{s}^L(q)\right)$ are such that $\hat{\Delta}(q) > 0$, $\hat{s}^H(q) > \bar{s}^H$ and $\hat{s}^L(q) > \bar{s}^L$.*

The existence and uniqueness of such an S.E. was shown by Matthews and Mirman (1983)) for stochastic signaling games in which the receiver faces a binary choice set. In the present setting, the proof of proposition 1 establishes first that for any equilibrium, both sender types signal more than their myopic optimum, i.e. $\hat{s}^H(q) > \bar{s}^H$ and $\hat{s}^L(q) > \bar{s}^L$ and that in any equilibrium $\hat{\Delta}(q) \geq 0$. The next step demonstrates that if the sender's problem is strictly concave for all strategy profiles, one can construct for each sender type a continuous function similar to best response functions in e.g. Cournot games, and that an S.E. is constituted by a crossing of these functions. For the high sender type, such a function indicates for each level of s^L the unique level of s^H which satisfies (11) for consistent beliefs (8) and (9) if this implies Δ, or $s^H = s^L$ if the constraint $\Delta \geq 0$ is binding (such that the marginal utility in (11) is strictly negative for all $s^H \geq s^L$). This function is strictly above $\bar{s}^H$ for $s^L = \bar{s}^L$, and equals s^L for s^L sufficiently high. A similar function for the low sender type is shown to take only values in $\left(\bar{s}^L, s^H\right)$. Note that this implies that both functions cross at least once, or that an S.E. exists. The uniqueness of such crossing is shown by an elementary application of the Poincaré-Hopf index theorem.

How does more accurate exogenous information affect equilibrium signaling? I impose an additional technical condition, which bounds the accuracy of exogenous information ω from above.

Condition 3 *Let* $q < \frac{2+\sqrt{3}}{4} \cong 0.933.$

The following theorem then shows that equilibrium signaling increases with the accuracy of exogenous information if prior beliefs are below a threshold.

Theorem 1 *If φ, v and q satisfy, respectively, conditions 1, 2 and 3 and if the sender's problem is strictly concave, then a unique threshold $\bar{p}(q)$ exists such that:*
if $p < \bar{p}(q)$, then $\hat{s}_1^H(q) > 0$ and $\hat{s}_1^L(q) > 0$,
if $p > \bar{p}(q)$, then $\hat{s}_1^H(q) < 0$ and $\hat{s}_1^L(q) < 0$.
Moreover, $\bar{p}(q)$ is a continuous function of q.

Hence, an interval of sufficiently low prior beliefs generically exists at which the equilibrium signaling of both sender types increases with the accuracy of exogenous information. Figure 1 displays a numerical solution of threshold $\bar{p}(q)$, for φ the normal density function at $\sigma = 2$ and for three values of Δ. In an S.E. below $\bar{p}(q)$, more accurate exogenous information induces both sender types to signal more, while the opposite is the case above $\bar{p}(q)$.

To understand theorem 1, note first that marginal changes in q affect the first order conditions in (11) and (12) only through $B'(\Delta)$ and in a very similar way, such that an increase in $B'(\Delta)$ necessarily increases the equilibrium signaling efforts of both sender types. If a marginal increase in accuracy q increases $B'(\Delta)$, both sender types are encouraged to signal more: the high type to establish more separation, the low type to undo more separation. A marginal increase in accuracy q affects $B'(\Delta)$ in two ways: by changing the probability density of exogenous signal ω and by changing the receiver's equilibrium beliefs:

$$B_3'(\Delta|p,q) = \int \left[\tilde{\beta}(y,H|p,q) - \tilde{\beta}(y,L|p,q)\right]\varphi_2(y|\Delta)\,dy \qquad (13)$$
$$+\int \left[q\tilde{\beta}_4(y,H|p,q) + (1-q)\,\tilde{\beta}_4(y,L|p,q)\right]\varphi_2(y|\Delta)\,dy.$$

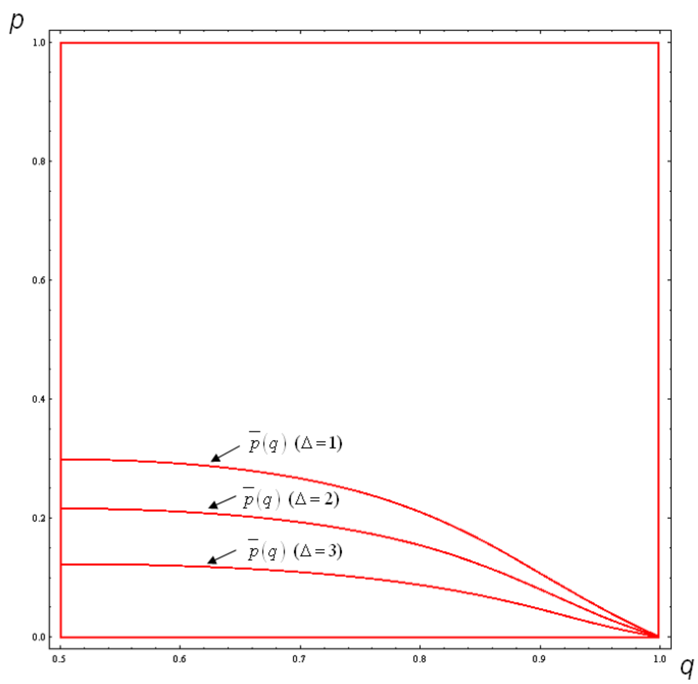

Figure 1: $\bar{p}(q)$ for φ the normal density function with $\sigma = 2$ at Δ equal to 1, 2 and 3.

In general, the sensitivity of B to Δ is greater if the receiver is more likely to see imperfect signals (y, ω) which she attributes to a high type with intermediate probability, while this effect is relatively small if the most probable imperfect imperfect signals come almost certainly from one sender type. The first integral in (13) is positive for p and Δ sufficiently low. In this case, for most y with a nontrivial probability mass under $\varphi(y|\Delta)$, the receiver's equilibrium beliefs are low if $\omega = L$ and (more) intermediate if $\omega = H$, such that drawing $\omega = H$ more often enhances the marginal effect of Δ. By the same reasoning, the first effect is negative for high p or Δ. The change in the receiver's equilibrium beliefs is represented in the second integral in (13), and resembles the effect of an increase in prior beliefs for $\omega = H$ and decrease in p if $\omega = L$. For $\omega = H$, the marginal effect of Δ increases for sufficiently low p or Δ, and decreases for sufficiently high p or Δ. If $\omega = L$, the opposite is the case. Bringing these partial effects together, theorem 1 states that a threshold $\bar{p}(q)$ exists, such that the equilibrium signaling increases with q for prior beliefs below $\bar{p}(q)$, and increases with q for p above $\bar{p}(q)$. Note that a higher Δ implies a lower $\bar{p}(q)$, as illustrated in figure 1: on average more informative y implies that only for the most pessimistic receivers more accurate exogenous information is needed to bring imperfect signals (y, ω) which the receiver attributes to a high sender type with an intermediate or high probability more within reach of the sender. By the same logic, $\bar{p}(q)$ can be seen to decrease with q.

The proof of theorem 1 shows first that $\hat{s}_1^H(q)$ and $\hat{s}_1^L(q)$ have the same sign as $B_3'(\Delta|p, q)$, then demonstrates that $B_3'(\Delta|p, q)$ is a continuous function of p, positive for p close to zero and negative for p close to one, and finally establishes that $B_3'(\Delta|p, q)$ strictly increases with p whenever it is zero. The method of proof limits the validity of this last step to q satisfying condition 3, although numerical solutions, as illustrated in Figure 1, suggest that theorem 1 is also true for higher q.

Contrary to Veblen's intuition and previous analyses, theorem 1 shows that more accurate exogenous information can push informed parties to waste more on costly signaling. In example 3, a monopolist is selling a new product, about which specialized media will publish product tests to distinguish between a true innovation and a marketing scam. If customers deem the chance of a true innovation sufficiently low, then an improved reliability of tests increases advertising by both true innovators and imitators selling junk. More reliable tests more often convince customers that a truly good product might indeed be a true innovation, whereas without these tests, the advertising needed to convince enough customers is prohibitively high. By distinguishing better between true innovators and imitators, more reliable tests also raise the stakes for the latter, who accordingly increase their equilibrium advertising to restore confusion with true innovators.

For the status signaling example, the literature often considers the benefits of information transfers at best as negligible in terms of welfare, as they merely reallocate status (rank, prestige) among consumers while status competition is a zero-sum game (see e.g. Frank 1985, 1999). Therefore, this literature is more concerned with abating conspicuous waste. Theorem 1 suggests that, contrary to some common intuitions, directly revealing information about individual qualities is not necessarily effective in reducing aggregate waste if this revelation is imperfect. If stronger social networks supply more accurate gossip information about individual quality, then contrary to Veblen's intuition stronger social networks can make all consumers waste more on status signaling if p is sufficiently low, e.g. because a community has only few high income consumers.

How does more accurate exogenous information help the receiver? A greater accuracy of exogenous information ω affects the expected accuracy of the receiver's equilibrium beliefs, as measured by B, in two ways: directly by providing more accurate ω, and indirectly by changing the average informativeness of equilibrium signaling:

$$B_q\left(\hat{\Delta}(q)\,|p,q\right) = \underbrace{B_3\left(\hat{\Delta}(q)\,|p,q\right)}_{\text{direct}} + \underbrace{B_1\left(\hat{\Delta}(q)\,|p,q\right)\hat{\Delta}_1(q)}_{\text{indirect}}.$$

The direct effect $B_3\left(\hat{\Delta}(q)\,|p,q\right)$ is always positive: given Δ, more accurate exogenous information improves the expected accuracy of the receiver's equilibrium beliefs. For the indirect effect, $B_1\left(\hat{\Delta}(q)\,|p,q\right) > 0$ by lemma 2: more separation in signaling helps the receiver in expectation to distinguish between sender types. Because a marginal increase in q affects the first order conditions in (11) and (12) similarly through $B'\left(\hat{\Delta}(q)\right)$, $\hat{s}_1^H(q)$ and $\hat{s}_1^L(q)$ have the same sign, but the relative size of both effects depends on p and the relative rate at which the signaling costs of both sender types increase at equilibrium. Define then

$$h\left(s^L, s^H\right) \equiv (1-p)\,v_{11}\left(s^L|\theta^L\right) - p v_{11}\left(s^H|\theta^H\right)$$

as the weighted difference in the rate at which the marginal utility costs of signaling increase for either sender type. The next result shows that an open interval of intermediate prior beliefs generically exists for which a marginal increase in q induces a decrease in the average informativeness of costly signaling, and that this decrease can come to dominate the direct information benefits of more accurate exogenous information.

Proposition 2 *If the conditions of theorem 1 apply, then:*

1. $\hat{\Delta}_1(q)$ *takes the opposite sign of* $\hat{s}_1^H(q)$ *if* $h\left(\hat{s}^L, \hat{s}^H\right) > 0$ *and the same sign as* $\hat{s}_1^H(q)$ *if* $h\left(\hat{s}^L, \hat{s}^H\right) < 0$,

2. *for* φ *the normal distribution,* $B_q\left(\hat{\Delta}(q)|p, q\right) < 0$ *for a non-empty part of parameter space.*

First, if the high type's signaling costs increase sufficiently more than the low type's, such that $h\left(s^L, s^H\right) > 0$, then the latter reacts more to changes in signaling incentives (i.e. in $B'(\Delta)$) than the former. In this case, more accurate exogenous information induces a less informative signaling equilibrium ($\hat{\Delta}_1(q) < 0$) if it enhances signaling incentives ($\hat{s}_1^H(q) > 0$), and a more informative signaling equilibrium if it diminishes equilibrium signaling. If $\hat{p}$ denotes the prior beliefs at which $h\left(s^L, s^H\right) = 0$,[11] then the first part of proposition 2 shows that $\hat{\Delta}_1(q) < 0$ only in an open interval between $\bar{p}$ and $\hat{p}$. Second, in this case the negative indirect effect can outweigh the positive direct effect on the expected accuracy of the receiver's equilibrium beliefs.

In example 2, the receiver is an employer estimating a job candidate's productivity from her educational achievements and through psychometric tests. Proposition 2 demonstrates that more accurate psychometric tests can induce a less informative signaling equilibrium, and in some case reduce the expected accuracy of the receiver's equilibrium beliefs. If psychometric tests become more widespread and reliable, if high productivity candidates are sufficiently prevalent and if low productivity types' signaling costs increase steeply, then job candidates rely more on psychometric tests and cut on education, and high productivity types reduce their education more than low types. In these cases, psychometric tests can make the employer worse off, by increasing the average mismatch between the wage and true productivity of her employees, even if the implementation of such psychometric tests were costless.

[11] That is: $\hat{p} = \frac{v_{11}\left(s^L|\theta^L\right)}{v_{11}\left(s^L|\theta^L\right) + v_{11}\left(s^H|\theta^H\right)}$.

4 Conclusions

The ostentatious waste associated with costly signaling is generally understood as a necessary cost for a transfer of information which otherwise cannot be credibly communicated. This paper developed a simple model of stochastic costly signaling in the presence of exogenous imperfect information, and studied how more accurate exogenous information affects the equilibrium signaling costs as well as the information eventually held in expectation by the receiver. Previous literature, mostly focussing on non-stochastic signaling with imperfect exogenous information, has found that better exogenous information can reduce equilibrium signaling, by offering the receiver more means to distinguish between the sender types. The present analysis demonstrates that more accurate exogenous information can generically both decrease and increase equilibrium costly signaling, depending on the receiver's prior beliefs. The intuition for the latter result is generic: for sufficiently pessimistic prior beliefs, the signaling levels required to generate with non-negligible likelihood noisy signals which the receiver attributes to a high sender type with intermediate or high probability are prohibitively high. More accurate exogenous information brings these noisy signals more likely within reach of high sender types, thus increasing their marginal benefits of signaling. More accurate exogenous information can also cause the receiver to be less well informed in equilibrium. More accurate exogenous information, although improving the receiver's information as a direct effect (i.e. for fixed signaling strategies), can by changing equilibrium signaling induce a decrease in the average informativeness of the distorted equilibrium signals. The latter effect of more accurate exogenous information can dominate the former, thus decreasing the expected accuracy of the receiver's equilibrium beliefs.

References

Akerlof, G. A. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 84(3), pp. 488–500.

Athey, S. (2002): "Monotone comparative statics under uncertainty," *Quarterly Journal of Economics*, 117(1), 187–223.

Bagwell, K. (2007): "The Economic Analysis of Advertising," in *Handbook of Industrial Organization*, ed. by M. Armstrong, and R. Porter, vol. 3 of *Handbook of Industrial Organization*, chap. 28, pp. 1701 – 1844. Elsevier.

Bhattacharya, S. (1979): "Imperfect Information, Dividend Policy, and "The Bird in the Hand" Fallacy," *The Bell Journal of Economics*, 10(1), pp. 259–270.

Blume, A., and O. Board (2009): "Intentional vagueness," *Erkenntnis*, pp. 1–45.

Blume, A., O. J. Board, and K. Kawamura (2007): "Noisy talk," *Theoretical Economics*, 2(4), 395–440.

Carlsson, H., and S. Dasgupta (1997): "Noise-Proof Equilibria in Two-Action Signaling Games," *Journal of Economic Theory*, 77, 432–460.

Chen, Y. (2012): "Value of public information in sender-receiver games," *Economics Letters*, 114(3), 343–345.

Chenault, L. A. (1986): "On the uniqueness of Nash equilibria," *Economics Letters*, 20(3), 203 – 205.

Crawford, V., and J. Sobel (1982): "Strategic information-transmission," *Econometrica*, 50(6), 1431–1451.

Daley, B., and B. Green (2012): "Market Signaling with Grades," *unpublished manuscript*.

de Haan, T., T. Offerman, and R. Sloof (2011): "Noisy Signaling: Theory and Experiment," *Games and Economic Behavior*, 73(2), 402–428.

Feltovich, N., R. Harbaugh, and T. To (2002): "Too cool for school? Signalling and countersignalling," *Rand Journal of Economics*, 33, 630–649.

FRANK, R. (1985a): *Choosing the Right Pond. Human Behavior and the Quest for Status*. Oxford University Press, New York, Oxford.

——— (1999): *Luxury Fever. Money and Happiness in an Era of Excess*. Princeton University Press, Princeton.

FRANK, R. H. (1985b): "The Demand for Unobservable and other Nonpositional Goods," *American Economic Review*, 75, 101–116.

GRAFEN, A. (1990a): "Biological Signals As Handicaps," *Journal of Theoretical Biology*, 144, 517–546.

——— (1990b): "Sexual Selection Unhandicapped by the Fisher Process," *Journal of Theoretical Biology*, 144, 473–516.

GUILLEMIN, V., AND A. POLLACK (1974): *Differential Topology*. Englewood Cliffs, N.J.: Prentice Hall.

HERTZENDORF, M. N. (1993): "Im Not A High-Quality Firm - But I Play One on Tv," *Rand Journal of Economics*, 24, 236–247.

JEITSCHKO, T. D., AND H.-T. NORMANN (2012): "Signaling in deterministic and stochastic settings," *Journal of Economic Behavior & Organization*, 82(1), 39 – 55.

JOHN, K., AND J. WILLIAMS (1985): "Dividends, Dilution, and Taxes: A Signalling Equilibrium," *The Journal of Finance*, 40(4), pp. 1053–1070.

KARLIN, S., AND H. RUBIN (1956): "The Theory of Decision Procedures for Distributions with Monotone Likelihood Ratio," *The Annals of Mathematical Statistics*, 27(2), 272–299.

MAILATH, G. (1987): "Incentive Compatibility in Signaling Games with a Continuum of Types," *Econometrica*, 55, 1349–1365.

MAILATH, G. J., M. OKUNOFUJIWARA, AND A. POSTLEWAITE (1993): "Belief-Based Refinements in Signaling Games," *Journal of Economic Theory*, 60, 241–276.

MATTHEWS, S. A., AND L. J. MIRMAN (1983): "Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand," *Econometrica*, 51(4), 981–96.

Milgrom, P., and J. Roberts (1986): "Price and Advertising Signals of Product Quality," *Journal of Political Economy*, 94(4), 796–821.

Myers, S. C., and N. S. Majluf (1984): "Corporate financing and investment decisions when firms have information that investors do not have," *Journal of Financial Economics*, 13(2), 187 – 221.

Riley, J. G. (1979): "Informational Equilibrium," *Econometrica*, 47, 331–359.

——— (2001): "Silver signals: Twenty-five years of screening and signaling," *Journal of Economic Literature*, 39, 432–478.

Spence, M. A. (1973): "Job Market Signaling," *Quarterly Journal of Economics*, 87, 355–374.

Truyts, T. (2010): "Social Status in Economic Theory," *Journal of Economic Surveys*, 24(1), 137–169.

——— (2012): "Signaling and Indirect Taxation," *Journal of Public Economics*, 96(3-4), 331–340.

Veblen, T. (1899): *The Theory of the Leisure Class: an economic study of institutions*. MacMillan, New York-London.

Zahavi, A. (1975): "Mate Selection - Selection for A Handicap," *Journal of Theoretical Biology*, 53, 205–214.

A Mathematical Appendix: Proofs

It will be convenient to write

$$c\left(y,q\right)\equiv q\tilde{\beta}\left(y,H\right)+\left(1-q\right)\tilde{\beta}\left(y,L\right),$$

and to denote

$$\begin{aligned} F^{H}\left(s^{H},s^{L}|q\right) &\equiv v_{1}\left(s^{H}|\theta^{H}\right)+\kappa B'\left(\Delta\right) \\ F^{L}\left(s^{L},s^{H}|q\right) &\equiv v_{1}\left(s^{L}|\theta^{L}\right)+\kappa\frac{p}{1-p}B'\left(\Delta\right). \end{aligned}$$

A.1 Proof of lemma 1

To see the first part, write

$$\begin{aligned} & p\bar{B}\left(s^H|\theta^H,\beta\right)+(1-p)\,\bar{B}\left(s^L|\theta^L,\beta\right) \\ =& \int\beta\left(y,H\right)\left[pq\varphi\left(y|s^H\right)+(1-q)\left(1-p\right)\varphi\left(y|s^L\right)\right]dy \\ &+\int\beta\left(y,L\right)\left[q\left(1-p\right)\varphi\left(y|s^L\right)+(1-q)\,p\varphi\left(y|s^H\right)\right]dy \\ =& pq+(1-q)\,p=p. \end{aligned}$$

The second part follows from the assumption that the distribution of ε is independent of s, such that $\varphi\left(y+a|s+a\right)=\varphi\left(y|s\right)$ for all $a\in\mathbb{R}$, and the integral in B being indefinite.

A.2 Proof of lemma 2

By condition 1, $\tilde{\beta}$ is strictly increasing with y if $\Delta>0$ and constant if $\Delta=0$, such that

$$\begin{aligned} B'\left(\Delta\right)=&\, q\int_{\Delta}^{+\infty}\left(\tilde{\beta}\left(y,H\right)-\tilde{\beta}\left(2\Delta-y,H\right)\right)\left|\varphi'(y|\Delta)\right|dy \\ &+(1-q)\int_{\Delta}^{+\infty}\left(\tilde{\beta}\left(y,L\right)-\tilde{\beta}\left(2\Delta-y,L\right)\right)\left|\varphi'(y|\Delta)\right|dy>0 \end{aligned} \tag{14}$$

if $\Delta>0$ and $B'\left(0\right)=0$. Next, use lemma 1 to write

$$B\left(\Delta\right)=1-\frac{1-p}{p}\int c\left(y,q\right)\varphi(y|0)dy,$$

such that

$$B_1\left(\Delta\right)=-\int\left(\frac{q^2}{1-q}\left(1-\tilde{\beta}\left(y,H\right)\right)^2+\frac{\left(1-q\right)^2}{q}\left(1-\tilde{\beta}\left(y,L\right)\right)^2\right)\varphi_2(y|\Delta)dy,$$

in which, by condition 1, $-\left(1-\tilde{\beta}\left(y,\omega\right)\right)^2$ is strictly increasing with y if $\Delta>0$ and constant if $\Delta=0$. Use condition 1 to write $B_1\left(\Delta\right)$ as an integral over $[\Delta,\infty)$, as in (14), to obtain $B_1\left(\Delta\right)>0$ for $\Delta>0$ and $B_1\left(0\right)=0$.

A.3 Proof of proposition 1

Define the second order derivative of B, taking $\tilde{\beta}$ as given,

$$B''(\Delta) \equiv \int c(y,q)\,\varphi_{22}(y|\Delta)dy,$$

while differentiating $B'(\Delta)$ to Δ (including the Δ in $\tilde{\beta}$) gives

$$B_1'(\Delta) = B''(\Delta) + \frac{p}{1-p}\int \begin{pmatrix} \frac{q^2}{1-q}\left(1-\tilde{\beta}(y,H)\right)^2 \\ +\frac{(1-q)^2}{q}\left(1-\tilde{\beta}(y,L)\right)^2 \end{pmatrix} \frac{(\varphi_2(y|\Delta))^2}{\varphi(y|0)}dy, \tag{15}$$

in which the second term is always positive. In general, $B_1'(\Delta)$ can be both positive and negative, such that condition 2 must be strengthened with an additional strict concavity condition.

Condition 4 *Let u and φ be such that for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:*

$$\begin{aligned} v_{11}\left(s^H|\theta^H\right) + \kappa B_1'(\Delta) &< 0 \\ v_{11}\left(s^L|\theta^L\right) - \kappa\frac{p}{1-p}B''(\Delta) &< 0. \end{aligned}$$

This condition encompasses two sets of second order conditions. First, a solution to (11) and (12) is a maximum for given beliefs (8) and (9) if for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:

$$v_{11}\left(s^H|\theta^H\right) + \kappa B''(\Delta) < 0 \tag{16}$$

$$v_{11}\left(s^L|\theta^L\right) - \kappa\frac{p}{1-p}B''(\Delta) < 0. \tag{17}$$

On the other hand, for a given level of signaling of the other type, an interior solution to (11) and (12) defines a unique interior level of signaling consistent with an S.E. if for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:

$$S^H\left(s^H\right) \equiv v_{11}\left(s^H|\theta^H\right) + \kappa B_1'(\Delta) < 0 \tag{18}$$

$$S^L\left(s^L\right) \equiv v_{11}\left(s^L|\theta^L\right) - \kappa\frac{p}{1-p}B_1'(\Delta) < 0. \tag{19}$$

Because the second term in (15) is always nonnegative, (16) is implied by (18) and (19) is implied by (17).

Proof. i) *Any S.E. strategy profile* $\left(\hat{s}^L(q), \hat{s}^H(q)\right)$ *must be above* $\left(\bar{s}^L, \bar{s}^H\right)$. Assume otherwise. First, if $\hat{s}^L(q) \le \hat{s}^H(q)$, then $B'\left(\hat{\Delta}(q)\right) \ge 0$ and either $v_1\left(\hat{s}^H(q)|\theta^H\right) > 0$ or $v_1\left(\hat{s}^L(q)|\theta^L\right) > 0$, such that (11) and (12) cannot both be satisfied. Second, if $\hat{s}^L(q) > \hat{s}^H(q)$ and $\hat{s}^L(q) \le \bar{s}^H$, then at $\hat{s}^H(q) = \hat{s}^L(q)$ we have $v_1\left(\hat{s}^H(q)|\theta^H\right) \ge 0$ and $B'(0) = 0$, which implies in combination with condition 4 that (11) cannot be satisfied. If $\hat{s}^L(q) > \hat{s}^H(q)$ and $\hat{s}^L(q) > \bar{s}^H$, then $v_1\left(\hat{s}^L(q)|\theta^L\right) < 0$ and $B'\left(\hat{\Delta}(q)\right) < 0$, such that (12) cannot be satisfied.
ii) *In any equilibrium,* $\hat{\Delta}(q) \ge 0$. If $\hat{\Delta}(q) < 0$ (and $\hat{s}^L(q) \ge \bar{s}^H$ by the previous point), then the low sender type can strictly improve herself by signaling less, because $v_1\left(\hat{s}^L(q)|\theta^L\right) < 0$ and $B'\left(\hat{\Delta}(q)\Delta\right) < 0$.
iii) *Existence of response functions.* Let $b^L\left(s^H\right)$ represent for each value of $s^H \ge \bar{s}^H$ the s^L for which (12) is satisfied. Let $b^H\left(s^L\right)$ represent for each value of s^L the unique value of s^H for which $F^H\left(s^H, s^L|q\right)\Delta = 0$, such that $b^H\left(s^L\right) \ge s^L$ satisfies (11) or $b^H\left(s^L\right) = s^L$ if the constraint $\Delta \ge 0$ is binding. These functions are well defined, as condition 4 implies that v_{11} is everywhere sufficiently concave and κ is sufficiently small to guarantee $\kappa < -\frac{\max_s\left\{v_{11}\left(s|\theta^H\right)\right\}}{\max_\Delta\left\{B_1'(\Delta)\right\}}$ and $\kappa < \frac{1-p}{p}\frac{\max_s\left\{v_{11}\left(s|\theta^L\right)\right\}}{\min_\Delta\{B''(\Delta)\}}$. One can easily verify that $\lim_{\Delta\to\infty} B'(\Delta) = 0$, such that $\lim_{\Delta\to\infty} B_1'(\Delta) = 0$ and $\lim_{\Delta\to\infty} B''(\Delta) = 0$. By conditions 1 and 2, b^L and b^H are continuously differentiable.
iv) *Existence of an S.E.* A crossing of $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$ constitutes an S.E. Note then that $b^L\left(s^H\right) \in \left(\bar{s}^L, s^H\right)$, because for $s^H \ge \bar{s}^H$ by construction $v_1\left(\bar{s}^L|\theta^L\right) = 0$ and $B'\left(s^H - \bar{s}^L\right) > 0$ while $v_1\left(s^H|\theta^L\right) < 0$ and $B'(0) = 0$ at $s^L = s^H$. On the other hand, $b^H\left(\bar{s}^L\right) > \bar{s}^H$ because $v_1\left(\bar{s}^H|\theta^H\right) = 0$ and $B'\left(\bar{s}^H - \bar{s}^L\right) > 0$ by construction. Moreover, a threshold $\zeta \ge \bar{s}^H$ exists such that $b^H\left(s^L\right) = s^L$ for all $s^L \ge \zeta$, because in this case $F_1^H\left(s, s^L|q\right) < 0$ for all $s > s^L$. This implies that $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$ cross at least once, and at such crossing $\Delta > 0$, $b^L\left(s^H\right) > \bar{s}^L$ and $b^H\left(s^L\right) > \bar{s}^H$.
v) *Uniqueness.* This is shown by an elementary instance of the Poincaré-

Hopf index theorem (Guillemin and Pollack, , p. 134), as exemplified in a.o. Chenault (1986). Construct the auxiliary function $d\left(s^L\right) \equiv b^H\left(s^L\right) - \left(b^L\right)^{-1}\left(s^L\right)$, measuring the distance between $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$. Then

$$\begin{aligned}
d_1\left(s^L\right) &= b_1^H\left(s^L\right) - \frac{1}{b_1^L\left(s^H\right)} = -\frac{F_2^H\left(s^H, s^L|q\right)}{S^H\left(s^H\right)} + \frac{S^L\left(s^L\right)}{F_2^L\left(s^L, s^H|q\right)} > 0 \\
&\Leftrightarrow F_2^H\left(s^H, s^L|q\right) F_2^L\left(s^L, s^H|q\right) - S^L\left(s^L\right) S^H\left(s^H\right) \\
&= -\frac{p}{1-p}\left(\kappa B_1'\left(\Delta\right)\right)^2 - S^L\left(s^L\right) S^H\left(s^H\right) < 0,
\end{aligned}$$

which is always satisfied under condition 4. Because $d\left(s^L\right)$ crosses 0 at most once, the S.E. is unique. ■

A.4 Proof of theorem 1

This proof proceeds in 3 steps.

Claim 1 *$\hat{s}_1^H\left(q\right)$ and $\hat{s}_1^L\left(q\right)$ have the same sign as $B_3'\left(\hat{\Delta}\left(q\right)|p,q\right)$.*

Claim 2 *$B'\left(\Delta|p,q\right)$ is continuously differentiable w.r.t. q for $q \in \left(\frac{1}{2}, \frac{2+\sqrt{3}}{4}\right)$. $B_3'\left(\Delta|p,q\right)$ is strictly positive for p sufficiently close to 0, and strictly negative for p sufficiently close to 1.*

Claim 3 *$B_3'\left(\Delta|p,q\right)$ is continuous w.r.t. p, and at the $\bar{p}\left(q\right)$ (where $B_3'\left(\Delta|\bar{p}\left(q\right),q\right) = 0$), it must be that $B_{23}'\left(\Delta|\bar{p}\left(q\right),q\right) < 0$.*

Claims 1, 2 and 3 together imply theorem 1.

A.4.1 Proof of claim 1

Proof. Write $F_q^H\left(\hat{s}^H\left(q\right), \hat{s}^L\left(q\right)|q\right) = 0$ and $F_q^L\left(\hat{s}^L\left(q\right), \hat{s}^H\left(q\right)|q\right) = 0$ as a system

$$A \cdot \begin{pmatrix} \hat{s}_1^L\left(q\right) \\ \hat{s}_1^H\left(q\right) \end{pmatrix} = \begin{pmatrix} -\kappa B_3'\left(\hat{\Delta}\left(q\right)|p,q\right) \\ -\kappa\frac{p}{(1-p)} B_3'\left(\hat{\Delta}\left(q\right)|p,q\right) \end{pmatrix}, \tag{20}$$

with

$$A = \begin{pmatrix} -\kappa B_1'\left(\hat{\Delta}\left(q\right)\right) & v_{11}\left(\hat{s}^H\left(q\right)|\theta^H\right) + \kappa B_1'\left(\hat{\Delta}\left(q\right)\right) \\ v_{11}\left(\hat{s}^L\left(q\right)|\theta^L\right) - \frac{\kappa p}{(1-p)} B_1'\left(\hat{\Delta}\left(q\right)\right) & \frac{\kappa p}{(1-p)} B_1'\left(\hat{\Delta}\left(q\right)\right) \end{pmatrix}.$$

System (20) has a unique solution if $|A| \neq 0$ everywhere, which is satisfied under condition 4 as:

$$|A| = -\frac{p}{(1-p)}\left(\kappa B_1'\left(\hat{\Delta}(q)\right)\right)^2 - S^L\left(\hat{s}^L(q)\right)S^H\left(\hat{s}^H(q)\right) < 0.$$

The system is solved for $\hat{s}_1^L(q)$ and $\hat{s}_1^H(q)$ by Cramer's rule, such that $\hat{s}_1^L(q) = \frac{\kappa B_3'\left(\hat{\Delta}(q)|p,q\right)v_{11}\left(\hat{s}^L(q)|\theta^L\right)}{|A|}$ and $\hat{s}_1^H(q) = \frac{\kappa\frac{p}{(1-p)}B_3'\left(\hat{\Delta}(q)|p,q\right)v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{|A|}$. This implies that $\hat{s}_1^L(q)$ and $\hat{s}_1^H(q)$ take the same sign as $B_3'\left(\hat{\Delta}(q)|p,q\right)$.
■

A.4.2 Proof of claim 2

It will be convenient to define $z(y,p) \equiv \frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}$, such that $z(y,p) \in \mathbb{R}^+$ and $z_1(y,p) < 0$. Whenever obvious, the arguments of z are omitted. Further, I denote $P^1(y,\omega) \equiv \left(1-\tilde{\beta}(y,\omega)\right)\tilde{\beta}(y,\omega)$ and $P^2(y,\omega) \equiv \left(1-\tilde{\beta}(y,\omega)\right)\left(\tilde{\beta}(y,\omega)\right)^2$.

Proof. Write $B_3'(\Delta|p,q) = \int f(z|p,q)\varphi_2(y|\Delta)\,dy$, with

$$\begin{aligned} f(z|p,q) \equiv c_2(y,q) &= \tilde{\beta}(y,H) + \frac{P^1(y,H)}{1-q} - \tilde{\beta}(y,L) - \frac{P^1(y,L)}{q} \\ &= \frac{z^2(2q-1)(z+1)}{\left((q+(1-q)z)((1-q)+qz)\right)^2}. \end{aligned}$$

Note that $f(z|p,q) > 0$ for all $z \in \mathbb{R}^+$. By condition 1, $f(z|p,q)$ is continuous and bounded, such that $B'(\Delta)$ is differentiable w.r.t. q. Moreover, it is easily verified that $f(0,q) = 0$ and $\lim_{z\to+\infty} f(z|p,q) = 0$. Furthermore, $f(z|p,q)$ has a unique extremum in terms of z, a maximum, because

$$f_1(z|p,q) = z(2q-1)\frac{\left(-z^3a + z^2(1-4a) + 3za + 2a\right)}{\left((q+(1-q)z)((1-q)+qz)\right)^3},$$

with $a \equiv q(1-q)$, has for $q \in \left(\frac{1}{2},1\right)$ and $z > 0$ a strictly positive denominator which is finite for finite z. Then $f_1(z|p,q) = 0$ only where $-z^3a+z^2(1-4a)+3za+2a = 0$, which has a unique real root because its discriminant is $\delta = -a\left(-1088a^3 + 564a^2 - 105a + 8\right) < 0$ for $q \in \left(\frac{1}{2},1\right)$. This root, denoted ξ, is strictly positive and finite for q bounded away from 1 (while only $q < \frac{2+\sqrt{3}}{4}$ is considered):

$$\xi = \frac{(1-4a)}{3a} + \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(X + \sqrt{-27a^2\delta}\right)} + \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(X - \sqrt{-27a^2\delta}\right)} > 0,$$

with

$$X \equiv \left(2(1-4a)^3 + 27a^2(1-4a) + 54a^3\right).$$

Hence, $c_2(y,q)$ is unimodal with a unique maximum at $y_\xi(p)$, which solves $z(y_\xi(p), p) = \xi$, such that $c_2(y,q)$ is strictly increasing with y for $y < y_\xi(p)$ and strictly decreasing with y for $y > y_\xi(p)$.

Note that ξ is independent of p, and that by taking p sufficiently close to 0, $f(z|p,q)$ is strictly increasing with y for almost all mass under $|\varphi_2(y|\Delta)|$ such that $B_3'(\Delta|p,q) > 0$. Similarly, for p sufficiently close to 1, $f(z|p,q)$ is strictly decreasing with y for almost all mass under $|\varphi_2(y|\Delta)|$ such that $B_3'(\Delta|p,q) < 0$. ■

A.4.3 Proof of claim 3

Proof. First, $B_{23}'(\Delta|p,q) = -\frac{1}{p(1-p)}\int (zf_1(z|p,q))\,\varphi_2(y|\Delta)\,dy$ exists everywhere because $zf_1(z|p,q)$ is continuous w.r.t. y and bounded for $q \in \left(\frac{1}{2}, \frac{2+\sqrt{3}}{4}\right)$. Note also that $\lim_{z\to 0} zf_1(z|p,q) = 0$ and $\lim_{z\to+\infty} zf_1(z|p,q) = 0$.

Consider then $B_{23}'(\Delta|p,q) = \int f_2(z|p,q)\,\varphi_2(y|\Delta)\,dy$ with $\tilde{\beta}_3(y,\omega|p,q) = \frac{P^1(y,\omega)}{p(1-p)}$ such that

$$\begin{aligned} p(1-p)\,f_2(z|p,q) &= P^1(y,H) - P^1(y,L) \\ &\quad + \frac{P^1(y,H) - 2P^2(y,H)}{1-q} - \frac{P^1(y,L) - 2P^2(y,L)}{q} \\ &= f(z|p,q) - g(z|p,q), \end{aligned}$$

with

$$\begin{aligned} g(z|p,q) &\equiv \left(\tilde{\beta}(y,H)\right)^2 - \left(\tilde{\beta}(y,L)\right)^2 + 2\left(\frac{P^2(y,H)}{1-q} - \frac{P^2(y,L)}{q}\right) \\ &= \frac{z^2(2q-1)(5q^2z^2 - 2q^2z - 3q^2 - 5qz^2 + 2qz + 3q + 2z^2 + z)}{\left((q+(1-q)z)((1-q)+qz)\right)^3}, \end{aligned}$$

such that

$$p\left(1-p\right)B_{23}'\left(\Delta|p,q\right)=B_{3}'\left(\Delta|p,q\right)-\int g\left(z|p,q\right)\varphi_{2}\left(y|\Delta\right)dy$$

Define

$$r\left(z\right)\equiv\frac{g\left(z|p,q\right)}{f\left(z|p,q\right)}=\frac{\left(5q^{2}z^{2}-2q^{2}z-3q^{2}-5qz^{2}+2qz+3q+2z^{2}+z\right)}{\left(\left(q+\left(1-q\right)z\right)\left(\left(1-q\right)+qz\right)\right)\left(z+1\right)}.$$

One can verify that

$$r_{1}\left(z\right)=-2(1-q)q\left(\left(q+\left(1-q\right)z\right)^{-2}+\left(q(z-1)+1\right)^{-2}\right)+\left(z+1\right)^{-2}<0 \tag{21}$$

for all $z\in\mathbb{R}^{+}$ if $q<\frac{2+\sqrt{3}}{4}\cong 0.93301$. At $\bar{p}$, we have by definition $B_{3}'\left(\Delta|\bar{p},q\right)=\int_{\Delta}^{+\infty}\left[-f\left(z\left(2\Delta-y,\bar{p}\right)|\bar{p},q\right)+f\left(z\left(y,\bar{p}\right)|\bar{p},q\right)\right]\left|\varphi_{2}\left(y|\Delta\right)\right|dy=0$, which implies for all $q<\frac{2+\sqrt{3}}{4}$ that

$$-\bar{p}\left(1-\bar{p}\right)B_{23}'\left(\Delta|\bar{p},q\right)=\int_{\Delta}^{+\infty}\begin{bmatrix}-g\left(z\left(2\Delta-y,\bar{p}\right)|\bar{p},q\right)\\+g\left(z\left(y,\bar{p}\right)|\bar{p},q\right)\end{bmatrix}\left|\varphi_{2}\left(y|\Delta\right)\right|dy>0,$$

because by (21) we have

$$-g\left(z\left(2\Delta-y,p\right)|p,q\right)+g\left(z\left(y,p\right)|p,q\right)>-f\left(z\left(2\Delta-y,p\right)|p,q\right)+f\left(z\left(y,p\right)|p,q\right)$$

for all $y\in\left(\Delta,\infty\right)$. Hence, $B_{23}'\left(\Delta|p,q\right)<0$ at $\bar{p}$, and this implies that $\bar{p}\left(q\right)$ is unique for all $q\in\left(\frac{1}{2},\frac{2+\sqrt{3}}{4}\right)$. The continuity of $\bar{p}\left(q\right)$ follows from the differentiability of $B_{3}'\left(\Delta|p,q\right)$ w.r.t. p. ■

A.5 Proof of proposition 2

First, from the proof of claim 1 we obtain $\hat{\Delta}_{1}\left(q\right)=\hat{s}_{1}^{H}\left(q\right)-\hat{s}_{1}^{L}\left(q\right)=\frac{\kappa B_{3}'(\Delta|p,q)h\left(\hat{s}^{L}(q),\hat{s}^{H}(q)\right)}{|A|(1-p)}$, which establishes the first part of proposition 2.

The second part of proposition 2 is shown by constructing a numerical example for which $B_{3}\left(\hat{\Delta}\left(q\right)|p,q\right)+B_{1}\left(\hat{\Delta}\left(q\right)|p,q\right)\hat{\Delta}_{1}\left(q\right)<0$. Consider $\hat{\Delta}\left(q\right)=\frac{3}{2}$, $p=0.9$, $q=0.91$ and φ the normal density function with $\sigma=2$. In this case, we seek to construct an S.E. where $\frac{\kappa h\left(\hat{s}^{L}(q),\hat{s}^{H}(q)\right)}{|A|(1-p)}<\frac{B_{3}\left(\hat{\Delta}(q)|p,q\right)}{B_{1}\left(\hat{\Delta}(q)|p,q\right)B_{3}'\left(\hat{\Delta}(q)|p,q\right)}\simeq -395.095$. For these parameter values we also have $\max_{\Delta}\{|B_1'(\Delta)|\} < 0.008052 \equiv C$, $\max_{\Delta}\{|B''(\Delta)|\} < 0.0104 \equiv D$ and $B''\left(\frac{3}{2}\right) \simeq 0.00612$. Choose a utility function for which $-v_{11}\left(.|\theta^H\right)$ is minimal at $\hat{s}^H(q)$ and for which $v\left(.|\theta^L\right)$ is sufficiently concave to guarantee $\frac{-v_{11}\left(.|\theta^L\right)}{-v_{11}\left(\hat{s}^H(q)|\theta^H\right)} > \frac{9D}{C} = 15.289$. Note that this implies $h\left(\hat{s}^L(q), \hat{s}^H(q)\right) < 0$ and $\hat{\Delta}_1(q) < 0$ and that condition 4 is satisfied if we choose

$$\kappa = \frac{-v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{C}.$$

Thus, we obtain

$$\begin{aligned}\frac{\kappa h\left(\hat{s}_1^L(q), \hat{s}_1^H(q)\right)}{|A|(1-p)} &= \left(\frac{-C}{\left(1-\frac{p}{(1-p)}\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)}\right)} + B_1'(\Delta)\right)^{-1} \\ &= \left(\frac{-0.008052}{\left(1-9\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)}\right)} + 0.00612\right)^{-1}\end{aligned}$$

which is smaller than -395.095 if $\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)} \lessapprox 7.720 \times 10^{-3}$, i.e. if $-v_{11}\left(\hat{s}^L(q)|\theta^L\right) > 129.53\left(-v_{11}\left(\hat{s}^H(q)|\theta^H\right)\right)$.

Finally, an S.E. is constructed which satisfies the above restrictions. First, $B'\left(\hat{\Delta}(q)\right) \simeq 0.0111$ for the given parameter values, such that (11) can be written $\frac{v_1\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^H(q)|\theta^H\right)} = \frac{0.0111}{0.008052} = 1.3748$ and likewise (12) becomes $\frac{v_1\left(\hat{s}^L(q)|\theta^L\right)}{v_{11}\left(\hat{s}^H(q)|\theta^H\right)} = 9\frac{(0.0110702)}{0.008052} = 12.374$. No other restrictions impede the construction of such a function v.