Dominance Invariant One-to-One Matching Problems

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Abstract

Solution concepts in social environments use either a direct or indirect dominance relationship, depending on whether it is assumed that agents are myopic or farsighted. Direct dominance implies indirect dominance, but not the reverse. Hence, the predicted outcomes when assuming myopic (direct) or farsighted (indirect) agents could be very different. In this paper, we characterize dominance invariant one-to-one matching problems when preferences are strict. That is, we obtain the conditions on preference profiles such that indirect dominance implies direct dominance in these problems and give them an intuitive interpretation. Whenever some of the conditions are not satisfied, it is important to know the kind of agents that are being investigated in order to use the appropriate stability concept. Furthermore, we characterize dominance invariant one-to-one matching problems having a non-empty core. Finally, we show that, if the core of a dominance invariant one-to-one matching problem is not empty, it contains a unique matching, the dominance invariant stable matching, in which all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

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1 Introduction

Solution concepts in social environments use either a direct or indirect dominance relationship, depending on whether it is assumed that agents are myopic or farsighted. In solution concepts like the core and the von Neumann-Morgenstern stable set agents are not farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations. These concepts are based on the direct dominance relation and neglect the destabilizing effect of indirect dominance relations introduced by Harsanyi (1974) and formalized by Chwe (1994). Based on the concept of indirect dominance, several solution concepts assume farsighted behavior of the agents in abstract social environments, coalition formation, network formation or matching models.[1] These solution concepts include, among others, the largest consistent set and the von Neumann-Morgenstern farsightedly stable set (Chwe, 1994), the farsighted core (Diamantoudi and Xue, 2003), the farsightedly stable set (Herings et al., 2010), the pairwise farsightedly stable set (Herings et al., 2009) and the path dominance core (Page and Wooders, 2009).

Direct dominance implies indirect dominance. However, indirect dominance does not imply direct dominance. For this reason, any solution concept may give different predictions when considering either myopic or farsighted agents. For instance, in coalition formation games with positive spillovers (e.g. cartel formation with Cournot competition and economies with pure public goods) Herings et al. (2010) and Mauleon and Vannetelbosch (2004) showed that the grand coalition is a farsightedly stable set, a von Neumann-Morgenstern farsightedly stable set and it always belongs to the largest consistent set. However, myopic stability concepts like the α-core, β-core or von Neumann-Morgenstern stable set, do not select the grand coalition as a stable outcome. Regarding the marriage problem, Ehlers (2007) characterized von Neumann-Morgenstern stable sets using a direct dominance relation, if such sets exist. He showed that these can be larger than the core. Mauleon et al. (2011), using a different direct domination relation from the one used by Ehlers (2007), and Chwe's (1994) definition of indirect dominance, showed the existence

[1] See for instance Greenberg (1990), Chwe (1994) and Xue (1998) about abstract social environments; Diamantoudi and Xue (2003), Mauleon and Vannetelbosch (2004), and Herings et al. (2010) about coalition formation; Dutta et al. (2005), Page et al. (2005), Herings et al. (2009), and Page and Wooders (2009) about network formation; and Mauleon et al. (2011) and Klaus et al. (2011) about matching models.

of and completely characterized the von Neumann-Morgenstern farsightedly stable sets: a set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton and belongs to the core. They also showed that the farsighted core, defined as the set of matchings that are not indirectly dominated by other matchings, can be empty.[2] On the contrary, in the formation of free trade networks, the global free trade network is a pairwise (myopically) stable network and also a pairwise farsightedly stable set in the model of Goyal and Joshi (2006) (see, Zhang et al., 2013).

Recently, some experimental evidence has been provided about the existence of both myopic and farsighted agents.[3] However, it is not obvious to know ex-ante the type of agents that one is facing. Then, an interesting question to investigate is whether there are situations in which one should not care about the kind of agents that are being considered or, in other words, if there are situations where the predicted outcomes do not depend on the kind of agents (myopic or farsighted) that are involved in the studied situation. In the present paper, we characterize *dominance invariant* one-to-one matching problems, i.e., one-to-one matching problems for which indirect dominance implies direct dominance when agents have strict preferences. As a consequence, in these kinds of problems, any solution concept based on direct or indirect dominance will give the same predictions. One-to-one matching problems (Gale and Shapley, 1962) represent situations in which a finite set of agents has to be partitioned into pairs and singletons. These problems are known as roommate problems and they include, as a particular case, the well-known marriage problems. Roommate problems are also a particular model of hedonic coalition formation (in which coalitions are restricted to have at most two agents) and of network formation (in which each agent is restricted to have at most one link).[4] Hence, roommate problems are a particularly interesting class of matching problems that lie in the intersection of network and coalition formation models. For this reason, by characterizing dominance invariant one-to-one matching problems, we provide the

[2] The farsighted core only exists when the core contains a unique matching and no other matching indirectly dominates the matching in the core.

[3] Kirchsteiger et al. (2013) have tested whether subjects behave myopically or farsightedly when forming a network. They have shown that behaviors consistent with farsightedness account for 75 percent of the individual observations, while only 6 percent of the individual observations are consistent with myopic behavior.

[4] See Bogomolnaia and Jackson (2002) and Jackson and Watts (2002).

basis that could be used for characterizing dominance invariant coalition formation problems and dominance invariant network formation problems. To the best of our knowledge, no characterization of dominance invariance has been provided up to now.

First, we characterize dominance invariant marriage problems (Theorem 1). A marriage problem is dominance invariant if and only if two conditions are satisfied. When two agents prefer to be matched to one another than being on their own, we say that these two agents are mutually acceptable. The first condition then states that mutually acceptable agents must prefer each other to any other agent (we interpret this condition as 'reciprocity'). The second condition states that if a man m (a woman w, respectively) considers a mutually acceptable woman (man) that is not his worst choice among the mutually acceptable ones, then this woman (man) cannot like any other man (woman) more than m (w). We subsequently give an alternative set of properties of agents' preferences that are necessary and sufficient for the marriage problem being dominance invariant (Proposition 1).

Second, we generalize the previous results and characterize dominance invariant roommate problems (Theorem 2). A roommate problem is dominance invariant if and only if two conditions are satisfied. The first condition coincides with the 'reciprocity' condition defined for the marriage problem. The second condition concerns the position of agents in each individual ranking of the set of mutually acceptable agents. Consider, for instance, agent i's ranking of her mutually acceptable agents. If agent k (in i's ranking but not in the last position) prefers another agent l more than i, then agent i must rank k and l in the last two positions of her ranking, with k more preferred than l. Informally, agent i penalizes these two agents: she penalizes agent k for not thinking highly of her and penalizes agent l out of jealousy. (We interpret this condition as 'extreme jealousy'.) Notice that, with this characterization of dominance invariant roommate problems, and also with the characterization of dominance invariant marriage problems, we provide two easy to verify conditions that tell us when one should care or not about the kind of agents (myopic or farsighted) that are being investigated. Whenever one of the two conditions is not satisfied, it is important to understand whether the agents under consideration are myopic or farsighted in order to use the appropriate stability concept.

We subsequently give some properties of agents' preferences in a roommate problem which is dominance invariant (Proposition 2) and we show (Proposition 3) that a roommate problem with three agents who prefer being matched to being unmatched is always dominance invariant. Such a problem may have an empty core from which we conclude that the notion of dominance invariance has nothing in common with well-known restrictions on preferences guaranteeing existence and/or uniqueness of the core in the roommate problem such as α-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al. 2001). As α-reducibility and weak top coalition property, dominance invariance is not a solution concept. Instead dominance invariance aims at differentiating roommate problems for which farsightedness could matter from those for which it does not matter. It is a property that guarantees the robustness of predictions of any solution concept.

Next, we focus on and characterize dominance invariant one-to-one matching problems with a non-empty core, or solvable dominance invariant one-to-one matching problems. We show (Proposition 4) that a dominance invariant one-to-one matching problem is solvable when there does not exist a structure in the preference profile called *ring*, formed by three agents such that the members of this ring prefer the other agents in the ring to any other agent outside the ring. It is a well-known result that marriage problems belong to this class. This allows us to state (Proposition 5) that, if it exists, the core of a dominance invariant one-to-one matching problem contains a unique matching, the dominance invariant stable matching, in which all agents who mutually *top rank* each other are matched to each other and all other agents are single.

The rest of the paper is organized as follows. Section 2 introduces one-to-one matching problems. Section 3 defines dominance invariant one-to-one matching problems and contains our main results. Section 4 analyzes the existence of the core in these problems and characterize it when it exists. Section 5 concludes.

2 One-to-one matching problems

A one-to-one matching problem, or roommate problem, is a pair (N, P) where N is a finite set of agents and P is a preference profile specifying for each agent $i \in N$ a strict preference ordering over N. That is, $P = \{P(1), ..., P(i), ..., P(n)\}$, where $P(i)$ is agent i's strict preference ordering over the agents in N including herself, which can be interpreted as the prospect of being alone. For instance, $P(i) = 1, 3, i, 2, ...$ indicates that agent i prefers agent 1 to agent 3 and she prefers to remain alone rather than get matched to anyone else. We denote by R the weak orders associated with P. We write $j \succ_i k$ if agent i strictly prefers j to k, $j \sim_i k$ if i is indifferent between j and k, and $j \succsim_i k$ if $j \succ_i k$ or $j \sim_i k$. A marriage problem is a roommate problem (N, P) where N is the union of two disjoint finite sets: a set of men $M = \{m_1, \ldots, m_r\}$, and a set of women, $W = \{w_1, \ldots, w_s\}$, where possibly $r \neq s$, and P is a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), \ldots, P(m_r), P(w_1), \ldots, P(w_s)\}$. That is, each man (woman) prefers being unmatched to be matched with any other agent in M (W, respectively). Along the paper, we consider the two domains of one-to-one matching problems: roommate and marriage problems. For the sake of notational simplicity, we use the more general domain of roommate problems in the definitions below.

A *matching* μ is a function $\mu : N \to N$ such that for all $i \in N$, if $\mu(i) = j$, then $\mu(j) = i$. Agent $\mu(i)$ is agent i's *mate* at μ; i.e., the agent with whom she is matched to (possibly herself). We denote by $\mathcal{M}$ the set of all matchings. A matching μ is *individually rational* if each agent is acceptable to his or her partner, i.e. $\mu(i) \succsim_i i$ for all $i \in N$. For a given matching μ, a pair $\{i, j\}$ (possibly $i = j$) is said to form a *blocking pair* if they are not matched to one another but prefer one another to their partner at μ, i.e. $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. A matching μ is *stable* if it is not blocked by any individual or any pair of agents. A roommate problem (N, P) is *solvable* if it has a stable matching. Otherwise, it is called *unsolvable*. Marriage problems belong to the class of solvable roommate problems (Gale and Shapley, 1962).

We extend each agent's preference over her potential partners to the set of matchings in the following way. We say that agent i prefers μ' to μ, if and only if agent i prefers her partner at μ' to her partner at μ, $\mu'(i) \succ_i \mu(i)$. Abusing notation, we write this as $\mu' \succ_i \mu$. A coalition S is a subset of N.[5] For $S \subseteq N$, $\mu(S) = \{\mu(i) : i \in S\}$ denotes the set of mates of agents in S at μ. A matching μ is *blocked by a coalition* $S \subseteq N$ if there exists a matching μ' such that $\mu'(S) = S$ and for all $i \in S$, $\mu' \succ_i \mu$. If S blocks μ, then S is called a *blocking coalition* for μ. Note that if a coalition $S \subseteq N$ blocks a matching μ, then there exists a pair $\{i, j\}$ (possibly $i = j$) that blocks μ. The *core* of a roommate problem, denoted by $C(N, P)$, consists of all matchings which are not blocked by any coalition. Note that for any roommate problem the set of stable matchings equals the core.

[5]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion.

Definition 1. Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold for any agent $i \in N$: (i) $\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and (ii) $\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \varnothing$.

In other words, this enforceability condition[6] implies both that any new pair in μ' that does not exist in μ must be formed by agents in S, and that in order to break an existing pair in μ, one of the two agents involved in that pair must belong to coalition S. Notice that the concept of enforceability is independent of preferences. Furthermore, the fact that coalition $S \subseteq N$ can enforce a matching μ' over μ implies that there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of disjoint pairs $\{i_0, j_0\}, ..., \{i_{K-1}, j_{K-1}\}$ (possibly for some $k \in \{0, 1, ..., K-1\}$, $i_k = j_k$) such that for any $k \in \{1, ..., K\}$, the pair $\{i_{k-1}, j_{k-1}\} \in S$ can enforce the matching μ^k over μ^{k-1}.

Definition 2. A matching μ is directly dominated by μ', denoted by $\mu < \mu'$, if there exists a coalition $S \subseteq N$ of agents such that $\mu' \succ_i \mu \; \forall i \in S$ and S can enforce μ' over μ.

An alternative way of defining the core of a roommate problem is by means of the dominance relation. A matching μ is in the core if there is no subset of agents who, by rearranging their partnerships only among themselves, possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. Formally, a matching μ is in the core if μ is not directly dominated by any other matching $\mu' \in \mathcal{M}$. Gale and Shapley (1962) showed that the core of a roommate problem may be empty.[7]

We now introduce the *indirect dominance* relation. A matching μ' *indirectly dominates* μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching μ' compared to the status-quo they face. Formally, indirect dominance is defined as follows.

Definition 3. A matching μ is indirectly dominated by μ', denoted by $\mu \ll \mu'$, if there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ such that for any $k \in \{1, ..., K\}$,

[6] This enforceability condition has also been used in Mauleon et al. (2011) and in Klaus et al. (2011).

[7] Several papers are devoted to analyzing the core as solution for this matching problem. See for instance Tan (1991), Chung (2000), Diamantoudi et al. (2004) and Iñarra et al. (2013).

(i) $\mu^K \succ_i \mu^{k-1}\ \forall i \in S^{k-1}$, and

(ii) coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}.

Direct dominance can be obtained directly from Definition 3 by setting $K = 1$. Obviously, if $\mu < \mu'$ then $\mu \ll \mu'$; i.e., direct dominance implies indirect dominance. Recently, Mauleon et al. (2011) have shown that, in marriage problems an individually rational matching μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ. Klaus et al. (2011) have generalized this result for roommate problems, and they have proved that an individually rational matching μ indirectly dominates another individually rational matching μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ.

Diamantoudi and Xue (2003) have shown that if a matching belongs to the core, then it indirectly dominates any other matching.

3 Dominance invariant one-to-one matching problems

We define a one-to-one matching problem to be dominance invariant if and only if indirect dominance implies direct dominance.

Definition 4. A one-to-one matching problem (N, P) is dominance invariant if the following condition holds:

$$\mu' \gg \mu \Leftrightarrow \mu' > \mu,\ \forall \mu, \mu' \in \mathcal{M}.$$

Let (N, P) be a one-to-one matching problem. Let $i \in N$. We denote by $t(i)$ the most preferred partner for agent i. That is, $t(i) \succsim_i j$ for any $j \in N$. Let T denote the set of agents who are ranked first by her most preferred agent; i.e.,

$$T = \{i \in N : \exists j \in N \text{ such that } j = t(i) \text{ and } i = t(j)\}.$$

Notice that if $i \in T$, then $t(t(i)) = i$.

Given the problem (N, P), the set A_i denotes the set of agents acceptable for agent i, that is $A_i = \{j \in N : j \succ_i i\}$ and the set M_i denotes the set of *mutually acceptable* agents for i, that is $M_i = \{j \in A_i : i \succ_j j\}$. Let $\omega(i) \in M_i$ denote the least preferred partner for i in this set; i.e., $\forall k \in M_i : k \succsim_i \omega(i)$. Let M_i^k denote the set of mutually acceptable agents of i who are *less preferred* than k, that is $M_i^k = \{j \in M_i : k \succ_i j\}$. Let $R, Z \subseteq N$ and let $i \in N$. If agent i strictly prefers every agent in R to any agent in Z, then agent i strictly prefers the set R to the set Z. This is denoted by $R \succ_i Z$.

3.1 Characterization of dominance invariant marriage problems

The following result characterizes the dominance invariant marriage problems.

Theorem 1. *A marriage problem (N, P) is dominance invariant if and only if the preference relation P satisfies the following two conditions. For all $m_i \in M$ ($w_i \in W$, respectively)*

(i) $M_{m_i} \succ_{m_i} A_{m_i} \setminus M_{m_i}$,

(ii) $\nexists\, w \in M_{m_i} \setminus \{\omega(m_i)\}$ *such that* $m_j \succ_w m_i$ *for any* $m_j \in M_w$.

The proof of this result, as well as all other proofs, may be found in Appendix B. The first condition can be seen as 'reciprocity', in the sense that man m_i prefers women that are mutually acceptable to him to women that do not accept him although he accepts them. The second condition says that if a man and a woman m_i, w are mutually acceptable, with $w \neq \omega(m_i)$, w cannot prefer another mutually acceptable man m_j more than m_i. This condition may be interpreted as "extreme jealousy". If man m_i likes woman w and vice versa, but woman w likes another mutually acceptable man m_j better (than man m_i), then w is the worst mutually acceptable woman for m_i.

The next proposition describes some properties of agents' preferences in a marriage problem which is dominance invariant.[8]

Proposition 1. *A marriage problem (N, P) such that for all $i \in N$, $M_i \succ_i A_i \setminus M_i$ is dominance invariance if and only if for all man $m_i \in M$ (for all $w_i \in W$, respectively) the following conditions on preferences are satisfied:*

a. *For all* $w_j \in M_{m_i} \setminus \{\omega(m_i)\}$, $t(w_j) = m_i$

b. *If* $|M_{\omega(m_i)}| \geq 2$, $\omega(m_i) \in T$, *otherwise* $t(\omega(m_i)) = m_i$.

[8]From now on, $|M|$ denotes the cardinality of the set M.

According to the properties in preferences of the previous proposition, for any man m_i (woman w_i, respectively), by (**a.**) $m_i \in T$. His least preferred mutually acceptable woman, $\omega(m_i)$, by (**b.**), either belongs to T if she has at least two mutually acceptable men or she considers m_i her top choice. The restrictions imposed by dominance invariances are indeed rather strict, but they allow for an intuitive interpretation. We will now show that dominance invariant roommate problems allow for some more leeway in terms of preferences: there can exist agents who mutually accept each other and do not belong to T, the set of agents who top rank each other.

3.2 Characterization of dominance invariant roommate problems

First, we introduce some additional notation and definitions that we need for characterizing dominance invariant roommate problems.

The notion of a *ring* is a key notion for the existence of stable matchings in roommate problems. A ring $S = \{s_1, ..., s_k\} \subseteq N$ is an ordered set of agents such that $k \geq 3$ and for all $i \in \{1, ..., k\}$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} s_i$ (subscript modulo k).

The existence of odd rings in the preference profile is a necessary condition for the emptiness of the core in a roommate problem. This is straightforward from the necessary and sufficient condition provided by Tan (1991) for the emptiness of the core in a roommate problem. We refer the reader to Appendix A for a compilation of definitions and results about the solvability of roommate problems.

Our main result characterizes the dominance invariant roommate problems.

Theorem 2. *A roommate problem (N, P) is dominance invariant if and only if the preference relation P satisfies the following two conditions for all $i \in N$:*

(i) $M_i \succ_i A_i \setminus M_i$,

(ii) if $\exists\, k \in M_i \setminus \{\omega(i)\}$ *and* $\exists\, l \in M_k$ *such that* $l \succ_k i$ *then* $M_i^k = \{l\}$.[9]

The first condition can be seen as 'reciprocity', as in the characterization of the marriage problem. The second condition says that if two agents i, k are mutually acceptable, with $k \neq \omega(i)$, but k prefers another mutually acceptable agent l more than i, then, there cannot be any agent mutually acceptable for i less preferred than

[9] Notice that l equals $\omega(i)$.

k, different from l. In other words, l is the least preferred potential partner for i among her mutually acceptable agents. Moreover, there are no agents in agent i's preferences less preferred than k but more preferred than l. Notice that, if $k = \omega(i)$, then condition (ii) holds by default. Condition (ii) only puts restrictions on preferences whenever $k \in M_i$ is different from $\omega(i)$ and k prefers some $l \in M_k$ to i. Therefore, if agent $k \in M_i$ and $l \succ_k i$ for some $l \in M_k$ either $k = \omega(i)$ or $M_i^k = \{l\}$. This condition may be interpreted as 'extreme jealousy'. If agent i likes agent k and vice versa, but agent k likes another mutually acceptable agent l better (than agent i), then either k is the worst mutually acceptable agent for i (agent i penalizes k for not considering her the best), or k and l are the worst ranked mutually acceptable agents for i, with l less preferred that k (agent i is jealous of k and l).

The next proposition describes some properties of agents' preferences in a roommate problem which is dominance invariant. They depend on the cardinality of the sets of mutually acceptable agents in the problem.

Proposition 2. *A roommate problem (N, P) such that for all $i \in N$, $M_i \succ_i A_i \setminus M_i$ is dominance invariance if and only if the following conditions on preferences are satisfied.*

P1. *For all agent i such that $|M_i| > 2$ let assume, without loss of generality, that $M_i = \{j_1, \ldots, j_k, \omega(i)\}$ such that $j_m \succ_i j_{m+1}$, $\forall m \in \{1, \ldots, k-1\}$ and $j_k \succ_i \omega(i)$. Then*

- ***a.*** $\forall j \in M_i \setminus \{j_k, \omega(i)\}$, $t(j) = i$,
- ***b.*** $t(j_k) \in \{i, \omega(i)\}$
  - ***b.1*** *If $t(j_k) = i$ then either $\omega(i) \in T$ or $t(\omega(i)) \in \{i, t(i)\}$, and*
  - ***b.2*** *If $t(j_k) = \omega(i)$ then $\omega(i) \in T$.*

P2. *For all agent i such that $|M_i| \leq 2$. Then either $t(i) \in T$ with $t(t(i)) \in \{i, \omega(i)\}$ or $i \in S$ where S is a ring in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

Let us interpret all these properties. Consider any agent i with more than two mutually acceptable agents. Then, by **P1.a**, $i \in T$. The least preferred agent for i, agent $\omega(i)$, by **P1.b** either belongs to T (ranking as top choice an agent $k \notin M_i$ (**b.1**) or j_k (**b.2**)) or ranks as top choice either i or $t(i)$ (**b.1**). The remaining agents in M_i rank i as their most preferred agent, except j_k when $t(\omega(i)) = j_k$. Consider now any agent i with at most two mutually acceptable agents. By **P2** either $i \in T$ or her top ranked agent is $j \in T$ with $t(j) \neq i$ or it belongs to a ring formed by 3 agents such that each player in the ring considers acceptable only the other agents in the ring. Properties **P1** and **P2** allow us to determine which agents belong to set T.

Corollary 1. *For all $i \notin T$, there is no agent $j \notin T$ such that $i \in M_j$, except for those belonging to a ring S in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

The previous corollary states that those agents who do not belong to set T cannot be mutually acceptable among them, except those belonging to a ring formed by 3 agents in which all agent in the ring prefer being matched among themselves to being matched with any other agent. This is the main difference with the marriage problem studied above.

Example 1. The following example of a dominance invariant roommate problem may be useful for clarifying the previous results. Agents that do not appear in the other agent's preferences are unacceptable.

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$P(7)$	$P(8)$
2	1	1	5	4	7	8	6
3	3	5	1	1	8	6	7
4	5	3	3	5	6	7	8
5	2		4				
1							

In this problem, the set of mutually acceptable agents are $M_1 = \{2, 3, 4, 5\}$, $M_2 = M_3 = \{1\}$, $M_4 = \{5, 1\}$ and $M_5 = \{4, 1\}$, $M_6 = \{7, 8\}$, $M_7 = \{6, 8\}$ and $M_8 = \{6, 7\}$. Notice that the first condition in Theorem 2 is satisfied since these agents are in the first rows of each agent's preferences. Consider for instance agent 1's preferences, $P(1)$. Notice that agents 1 and 4 are mutually acceptable and 4 is not the worse agent in M_1, however, $5 \succ_4 1$. Then, by condition (ii) of Theorem 2, agent 5 must be the immediate less preferred agent than 4 for agent 1. Notice that {6,7,8} form an odd ring in the preferences.

In this example, the only agent satisfying $|M_i| \geq 2$ is agent 1 with $M_1 = \{2, 3, 4, 5\}$ and $2 \succ_1 3 \succ_1 4 \succ_1 5$. We can see that $\forall j \in \{2, 3\}$, $t(j) = 1$ (**P1.a** in Proposition 2). Moreover, it must happen that $t(4) \in \{1, 5\}$ (**P1.b** in Proposition 2). In this case, $t(4) = 5$ and therefore $t(5) = 4$ (**P1.b.2** in Proposition 2).

On the other hand, all the other agents satisfy $|M_i| \leq 2$. For $i \in \{2, 3, 4, 5\}$, we can check that $t(i) \in T$. Agents in the set $\{6, 7, 8\}$ form a ring satisfying that $\forall s_i \in \{6, 7, 8\}$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} k$ for all $k \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Notice also that for $i \in \{1, 2, 3, 4, 5\}$, there is no pair of agents who do not belong to T such that they are mutually acceptable. In our example, the only agent who is not in T is agent 3, and there is no agent j in $P(3)$ ($j \notin T$) such that $j \succ_3 3$ and $j \in M_3$. □

The following result shows that all roommate problems such that $|N| = 3$ in which all agents prefer to be matched to being unmatched are dominance invariant.

Proposition 3. *Let (N, P) be a roommate problem such that $|N| = 3$ and $\forall i \in N$: $j \succ_i i$ for any $j \neq i$. Then (N, P) is dominance invariant.*

Note that this class of roommate problems can have an empty core when the three agents form an odd ring in P. This then implies that the notion of dominance invariance has little in common with restrictions on preferences which guarantee the existence and/or uniqueness of stable matchings (e.g. α-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al., 2001)). Dominance invariance is a property that differentiates roommate problems for which farsightedness matter from those for which it does not matter. It then guarantees the robustness of predictions of any solution concept.

4 Dominance invariance and the core

The following proposition characterizes the dominance invariant one-to-one matching problems with a non-empty core.

Proposition 4. *Let (N, P) be a dominance invariant one-to-one matching problem. $C(N, P) \neq \varnothing$ if and only if there is no ring S in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

Marriage problems are one-to-one matching problems with a non-empty core. This was already shown by Gale and Shapley (1962), however, it can be also deduced immediately from the previous proposition, given that in marriage problems there is no odd ring in preferences.

The following result, derived from the previous one, states that if a one-to-one matching problem is dominance invariant and the core is non-empty, it has a unique stable matching. In this stable matching, all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

Proposition 5. *Let (N, P) be a solvable dominance invariant one-to-one matching problem. Then, $C(N, P) = \{\mu_C\}$, where μ_C is such that $\mu_C(i) = t(i)$ for all $i \in T$, and $\mu_C(j) = j$ for all $j \notin T$.*

Example 1 (cont.) In this example, we have already seen that there is a ring $S = \{6, 7, 8\}$ in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$. Therefore this roommate problem is unsolvable and there is no stable matching.

Consider the problem derived from the previous one such that $N = \{1, 2, 3, 4, 5\}$ and $P = \{P(1), P(2), P(3), P(4), P(5)\}$. In this case, there is no ring in preferences satisfying the conditions above and therefore the problem is solvable. The core, in this case, is formed by the matching $\mu_C = \{\{1, 2\}, \{3\}, \{4, 5\}\}$. $\square$

Finally notice that, when the dominance invariant one-to-one matching problem is solvable, the core and the farsighted core (as well as the von Neumann-Morgenstern stable set defined by the direct domination and the von Neumann-Morgenstern farsightedly stable set) coincide.

5 Conclusion

We have characterized dominance invariant one-to-one matching problems when preferences are strict. That is, we have obtained under which conditions on preference profiles indirect dominance implies direct dominance in such problems. Hence, we have concluded that, whenever some of the conditions are not satisfied, one should try to know the kind of agents under consideration in order to use the appropriate stability concept. Furthermore, we have characterized solvable dominance invariant one-to-one matching problems. This characterization has allowed us to state that the core of such a problem contains a unique matching in which all agents who mutually top rank each other are matched to one another and all other agents remain unmatched.

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Appendix A

Tan (1991) establishes a necessary and sufficient condition for the solvability of roommate problems with strict preferences in terms of stable partitions. This notion, which is crucial in the investigation of the core for these problems, can be formally defined as follows.

Let $A = \{a_1, ..., a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, ..., k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k). The set A is a *pair of mutually acceptable agents* if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2).[10] The set A is a *singleton* if $k = 1$.

Definition 5. A stable partition is a partition P of N such that:
(i) for all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
(ii) for any sets $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if } \; b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

for all $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$ such that $b_j \neq a_{i+1}$.

[10] Hereafter we omit subscript modulo k.

Condition (i) specifies the sets contained in a stable partition, and condition (ii) contains the notion of stability to be applied between these sets (and also inside each set).

Note that a stable partition is a generalization of a stable matching. To see this, consider a matching μ and a partition P formed by pairs of agents and/or singletons. Let $A = \{a_1, a_2 = \mu(a_1)\}$ and $B = \{b_1, b_2 = \mu(b_1)\}$ be sets of P. If P is a stable partition then Condition (ii) implies that if $b_1 \succ_{a_2} a_1$ then $b_2 \succ_{b_1} a_2$, which is the usual notion of stability. Hence μ is a stable matching.

Remark 1 (Iñarra et al., 2010). (i) A roommate problem (N, P) has no stable matchings if and only if there exists a stable partition with an odd ring.[11] (ii) Any two stable partitions have exactly the same odd rings. (iii) Every even ring in a stable partition can be broken into pairs of mutually acceptable agents preserving stability.

Appendix B

Proof. [**Proof of Theorem 1**]

($\Rightarrow$) By contradiction, we will show that if one of the conditions (i) or (ii) is not satisfied, then $\mu \gg \mu' \nRightarrow \mu > \mu'$.

- Suppose that condition (i) is not satisfied and there exists a man $m_i \in M$ such that $w_k \succ_{m_i} w_j$ for some $w_k \in A_{m_i} \setminus M_{m_i}$ and some $w_j \in M_{m_i}$. Let μ_2 be a matching such that $\mu_2(m_i) = w_k$ and $\mu_2(s) = s$ for every $s \neq m_i, w_k$, and let μ_1 be a matching such that $\mu_1(m_i) = w_j$ and $\mu_1(s) = s$ for every $s \neq m_i, w_j$. Then $\mu_1 \gg \mu_2$ (since $w_k \succ_{w_k} m_i$, woman w_k enforces the matching in which every agent is alone, and this matching is blocked by $\{m_i, w_j\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2 \succ_{m_i} \mu_1$. A similar argument can be followed to show that there cannot be a woman $w_i \in W$ such that $m_k \succ_{w_i} m_j$ for some $m_k \in A_{w_i} \setminus M_{w_i}$ and some $m_j \in M_{w_i}$

- Suppose that condition (ii) is not satisfied and there exists a woman $w \in M_{m_i} \setminus \{\omega(m_i)\}$ such that $m_j \succ_w m_i$ for some $m_j \in M_w$. Let μ_2 be a matching such that $\mu_2(m_i) = w$ and $\mu_2(s) = s$ for every $s \neq m_i, w$, and let μ_1 be a matching such that $\mu_1(w) = m_j$, $\mu_1(m_i) = \omega(m_i)$ and where $\mu_1(s) = s$ for

[11] A ring is odd (even) if its cardinality is odd (even).

every $s \neq m_i, m_j, w, \omega(m_i)$. Then $\mu_1 \gg \mu_2$ (since $\{m_j, w\}$ block μ_2 enforcing a matching in which m_i and $\omega(m_i)$ are alone, and this matching is blocked by $\{m_i, \omega(m_i)\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2 \succ_{m_i} \mu_1$. In a similar way, we can show that there cannot be a man $m \in M_{w_i} \setminus \{\omega(w_i)\}$ such that $w_j \succ_m w_i$ for some $w_j \in M_m$

($\Leftarrow$) Now we will prove that if $\mu_1 \gg \mu_2$ and conditions (i) and (ii) are satisfied, then $\mu_1 > \mu_2$.

Let $D = \{i \in M \cup W : \mu_1 \succ_i \mu_2\}$. First, we prove that for any man $m_i \in M$ ($w_i \in W$, respectively) such that $\mu_1(m_i) \neq \mu_2(m_i)$ and $\mu_1(m_i) \neq m_i$, we have that $m_i \in D$. By contradiction, let $\mu_1(m_i) = w_j$ and let $\mu_2(m_i) = w_k$ and assume that $w_k \succ_{m_i} w_j$ (which implies that $w_k \neq w_j$). Notice that this implies that man m_i must have been left alone first and then matched to w_j, so $w_j \in M_{m_i}$ because otherwise $\{m_i, w_j\}$ would never be formed contradicting $\mu_1 \gg \mu_2$. Then, by condition (i), $w_k \in M_{m_i}$. Since $\mu_1 \gg \mu_2$ and m_i prefers μ_2 to μ_1, w_k must prefer μ_1 to μ_2 because, otherwise, $\{m_i, w_k\}$ would be a blocking pair of μ_1 contradicting that $\mu_1 \gg \mu_2$ [see Lemma 1 in Mauleon et al. (2011)].[12] Then, the partner of w_k at μ_1, say for instance $\mu_1(w_k) = m_l$, by condition (i), also belongs to the set M_{w_k} given that $m_l \succ_{w_k} m_i$, that is $m_l \in M_{w_k}$. But this contradicts condition (ii), which says if $w_k \in M_{m_i} \backslash \omega(m_i)$, there is no man $m_l \in M_{w_k}$ such that $m_l \succ_{w_k} m_i$. Hence, man m_i should prefer μ_1 to μ_2, and, by the same reasoning, $\mu_1 \succ_{w_j} \mu_2$. Therefore $\{m_i, \mu_1(m_i)\} \subseteq D$.

Consider now any man $m_i \in M$ such that $\mu_1(m_i) = m_i \neq \mu_2(m_i) = w_k$. Since $\mu_1 \gg \mu_2$, then either $\mu_1(w_k) \succ_{w_k} m_i$ and w_k deviates leaving agent m_i unmatched (with $\mu_1(w_k)$ also preferring μ_1 to μ_2) and then $\{w_k, \mu_1(w_k)\} \subseteq D$, or $m_i \succ_{m_i} w_k$ (and man m_i individually deviates) and therefore $m_i \in D$.

Then the coalition D deviates from μ_2 enforcing μ_1 and $\mu_1 > \mu_2$ as we wanted to prove. □

Proof. [**Proof of Proposition 1**]

Proof. ($\Leftarrow$) It is easy to see that if properties (**a.**) and (**b.**) are satisfied for every $m \in M$ and every $w \in W$, then condition (ii) of Theorem 1 is also satisfied and the problem (N, P) is dominance invariant. If condition (**a.**) holds for man $m_i \in M$,

[12] Although Lemma 1 in Mauleon et al. (2011) is stated requiring μ_1 to be individually rational, the "$\Rightarrow$"-part of the lemma, which is the one used in this proof, holds for any two different matchings (individually rational or not). See proof in Mauleon et al. (2011) (p. 515).

there is no woman $w_j \in M_{m_i} \setminus \{\omega(m_i)\}$ such that $m_j \succ_{w_j} m_i$ for any $m_j \in M_{w_j}$. If $|M_{\omega(m_i)}| \geq 2$ and condition (**b.**) holds, condition (ii) of Theorem 1 is satisfied by $\omega(m_i)$. Otherwise, $M_{\omega(m_i)} = \{m_i\}$ and condition (ii) of Theorem 1 is satisfied by default.

($\Rightarrow$) Now we show that if the problem is dominance invariant, that is, it satisfies conditions (i) and (ii) of Theorem 1, then properties (**a.**), and (**b.**) hold.

Suppose that condition (**a.**) is not satisfied. Then there is a woman $w_j \in M_{m_i} \setminus \{\omega(m_i)\}$ such that $t(w_j) \neq m_i$. Let $t(w_j) = m_j$. By condition (i) of Theorem 1 since $m_i \in M_{w_j}$ and $m_j \succ_{w_j} m_i$, we have that $m_j \in M_{w_j}$. However, this contradicts condition (ii) of Theorem 1.

Now we prove that property (**b.**) is also satisfied. Suppose first that $\omega(m_i) \notin T$, that is $t(t(\omega(m_i))) \neq \omega(m_i)$. If $t(\omega(m_i)) = m_i$ we are done, so assume that $t(\omega(m_i)) = m_j$ and $t(m_j) = w_k$. Then there is a man $m_j \in M_{\omega(m_i)} \setminus \{\omega(\omega(m_i))\}$ who prefers w_k to $\omega(m_i)$, contradicting that $\omega(m_i)$ satisfies condition (ii) of Theorem 1. Hence, if $\omega(m_i) \notin T$, the most prefer man for $\omega(m_i)$ is m_i. Suppose now that $t(\omega(m_i)) \neq m_i$, say $t(\omega(m_i)) = m_j$. By condition (i) of Theorem 1, since $m_i \in M_{\omega(m_i)}$ and $m_j \succ_{\omega(m_i)} m_i$, we have that $m_j \in M_{\omega(m_i)}$. By condition (ii) of Theorem 1 there cannot be any woman w_k such that $w_k \succ_{m_j} \omega(m_i)$ and therefore $t(m_j) = \omega(m_i)$ and $\omega(m_i) \in T$ as we wanted to prove. □

Proof. [**Proof of Theorem 2**]

($\Rightarrow$) By contradiction, we will show that if one of the conditions (i) or (ii) is not satisfied, then $\mu \gg \mu' \nRightarrow \mu > \mu'$.

- Suppose that condition (i) is not satisfied. Then there exists an agent $i \in N$ such that $k \succ_i j$ for some $k \in A_i \setminus M_i$ and some $j \in M_i$. Let μ_2 be a matching such that $\mu_2(i) = k$ and $\mu_2(s) = s$ for every $s \neq i, k$, and let μ_1 be a matching such that $\mu_1(i) = j$ and $\mu_1(s) = s$ for every $s \neq i, j$. Then $\mu_1 \gg \mu_2$ (since $k \succ_k i$, agent k enforces the matching in which every agent is alone, and this matching is blocked by $\{i, j\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2 \succ_i \mu_1$.
- Suppose that condition (ii) is not satisfied. Then there exists an agent $k \in M_i \setminus \{\omega(i)\}$ and an agent $l \in M_k$ such that $l \succ_k i$ and $\{l\} \neq M_i^k$. Then it must exist an agent $j \neq l$ such that $j \in M_i^k$. Let μ_2 be a matching such that $\mu_2(i) = k$ and $\mu_2(s) = s$ for every $s \neq i, k$, and let μ_1 be a matching such that $\mu_1(k) = l$, $\mu_1(i) = j$ and where $\mu_1(s) = s$ for every $s \neq i, k, l, j$. Then $\mu_1 \gg \mu_2$

(since $\{k,l\}$ block μ_2 enforcing a matching in which i and j are alone, and this matching is blocked by $\{i,j\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2 \succ_i \mu_1$.

($\Leftarrow$) Now we will prove that if $\mu_1 \gg \mu_2$ and conditions (i) and (ii) are satisfied, then $\mu_1 > \mu_2$.

Let $D = \{i \in N : \mu_1 \succ_i \mu_2\}$. First, we prove that for any agent $i \in N$ such that $\mu_1(i) \neq \mu_2(i)$ and $\mu_1(i) \neq i$, we have that $i \in D$. By contradiction, let $\mu_1(i) = j$ and let $\mu_2(i) = k$ and assume that $k \succ_i j$ (which implies that $k \neq j$). Notice that this implies that agent i must have been left alone first and then matched to j, so $j \in M_i$ because otherwise $\{i,j\}$ would never be formed contradicting $\mu_1 \gg \mu_2$. Then, by condition (i), $k \in M_i$. Since $\mu_1 \gg \mu_2$ and i prefers μ_2 to μ_1, k must prefer μ_1 to μ_2 because, otherwise, $\{i,k\}$ would be a blocking pair of μ_1 contradicting that $\mu_1 \gg \mu_2$ [see Proposition 1 in Klaus et al. (2011)].[13] Then, the partner of k at μ_1, say for instance $\mu_1(k) = l$ (with $l \neq k, j$), by condition (i), also belongs to the set of mutually acceptable agents of agent k given that $l \succ_k i$, that is $l \in M_k$. But then, according to (ii), it must be that $M_i^k = \{l\}$. But this is a contradiction, since $j \in M_i^k$. Hence, agent i should prefer $\mu_1(i)$ to $\mu_2(i)$, and, by the same reasoning, $\mu_1 \succ_j \mu_2$. Therefore $\{i, \mu_1(i)\} \subseteq D$.

Consider now any agent $i \in N$ such that $\mu_1(i) = i \neq \mu_2(i) = k$. Since $\mu_1 \gg \mu_2$, then either $\mu_1(k) \succ_k i$ and k deviates leaving agent i unmatched (with $\mu_1(k)$ also preferring μ_1 to μ_2) and then $\{k, \mu_1(k)\} \subseteq D$, or $i \succ_i k$ and agent i individually deviates and therefore $i \in D$.

Then the coalition D deviates from μ_2 enforcing μ_1 and $\mu_1 > \mu_2$ as we wanted to prove. □

Proof. [**Proof of Proposition 2**]

Proof. ($\Leftarrow$) It is easy to see that if properties [**P1.**], [**P2.**] are satisfied, then condition (ii) of Theorem 2 is also satisfied and the problem (N, P) is dominance invariant.

P1. If (**a**) and (**b.1**) are satisfied then there is no agent $k \in M_i \setminus \{\omega(i)\}$ such that $l \succ_k i$ for any $l \in M_k$. If (**b.2**) is satisfied then $j_k \in M_i \setminus \{\omega(i)\}$, $\omega(i) \succ_{j_k} i$, and $M_i^{j_k} = \{\omega(i)\}$. Hence, when agent i holds property [**P1.**] in her preferences, condition (ii) of Theorem 2 is satisfied.

[13] Although Proposition 1 in Klaus et al. (2011) is stated requiring μ and μ' to be individually rational, the "⇒"-part of the proposition, which is the one used in this proof, holds for any two different matchings (individually rational or not). See proof in Klaus et al. (2011) (pp. 926-927).

P2. If $M_i = \{j\}$, then $M_i \setminus \{\omega(i)\} = \emptyset$ and condition (ii) of Theorem 2 is satisfied by agent i by default. Let $M_i = \{j, \omega(i)\}$. If $t(t(i)) = i$ there is no agent $k \in M_i \setminus \{\omega(i)\}$ such that $l \succ_k i$ for any $l \in M_k$. If $t(t(i)) = \omega(i)$, then $t(i) \in M_i \setminus \{\omega(i)\}$, $\omega(i) \succ_{t(i)} i$, and $M_i^{t(i)} = \{\omega(i)\}$. In both cases, condition (ii) of Theorem 2 is satisfied by agent i. If $i \in S$ where S is a ring in P such that $|S| = 3$ and for all $s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$, then it is easy to see that condition (ii) of Theorem 2 is also satisfied by agent i. (Notice that if $t(t(i)) \notin \{i, \omega(i)\}$, condition (ii) is not satisfied and the problem is not dominance invariant)

($\Rightarrow$) We will show that if the problem is dominance invariant and satisfies conditions (i) and (ii) of Theorem 2, then properties [**P1.**], [**P2.**] hold.

P1. Assume that (**a**) is not satisfied. That is, there exists an agent $j \in M_i \setminus \{j_k, \omega(i)\}$ such that $t(j) \neq i$. This implies that $\exists k \in N$ such that $t(j) = k$ and $k \succ_j i$. By condition (i) of Theorem 2, $k \in M_j$ and then by condition (ii) of Theorem 2, $M_i^j = \{k\}$. However, this contradicts that $\{j_k, \omega(i)\} \subseteq M_i^j$.

Now we will show that (**b**) must be satisfied as well. The fact that $t(j_k) \in \{i, \omega(i)\}$ is straightforward from condition (ii) of Theorem 2.

In order to prove (**b.1**), let $t(j_k) = i$. First, we will show that if $\omega(i) \notin T$ then either $t(\omega(i)) = i$ or $t(\omega(i)) = t(i)$. Let $\omega(i) \notin T$. Then, there exists an agent k such that $t(\omega(i)) = k$. If $k = i$ we are done, so assume that $k \neq i$. Since $k \succ_{\omega(i)} i$ and $i \in M_{\omega(i)}$, by condition (i) of Theorem 2, $k \in M_{\omega(i)}$. Moreover, $t(k) = l \neq \omega(i)$ so $l \succ_k \omega(i)$ and by condition (i) of Theorem 2, $l \in M_k$. Thus, $\exists\, k \in M_{\omega(i)} \setminus \{\omega(\omega(i))\}$ and $\exists\, l \in M_k$ such that $l \succ_k \omega(i)$. Then, by condition (ii) of Theorem 2, it holds that $\{l\} = M^k_{\omega(i)}$. Since $i \in M^k_{\omega(i)}$, we have that $l = i$. Given that $l \in M_k$ and $l = i$, it holds that $k \in M_i$. Let $k \neq t(i)$, otherwise we are done. Then since $i \in M_k \setminus \{\omega(k)\}$ and there exists an agent $k' \in M_i$ such that $k' \succ_i k$ (remember that $k \neq t(i)$), by condition (ii) of Theorem 2, $\{k'\} = M_k^i$. But this implies that $k' = \omega(i)$ and this is a contradiction since $\omega(i) \not\succ_i k$. So we have proved that when $\omega(i) \notin T$ either $t(\omega(i)) = i$ or $t(\omega(i)) = t(i)$.

Now we will show that if $t(\omega(i)) \notin \{i, t(i)\}$, then $\omega(i) \in T$. Let $t(\omega(i)) = k$ with $k \neq i, t(i)$. If $t(k) = \omega(i)$ we are done, so assume that $t(k) = l \neq \omega(i)$. Thus, there exists an agent $l \in M_k$ such that $l \succ_k \omega(i)$ and by condition (ii) of Theorem 2 $\{l\} = M^k_{\omega(i)}$, which implies that $l = i$. Notice that we are in the same situation as in the previous paragraph. Then, following the same reasoning, we achieve the same contradiction ($\omega(i) \not\succ_i k$) and this proves that $\omega(i) \in T$ as desired.

Next, we proceed to prove (**b.2**). Let $t(j_k) = \omega(i)$. We will prove that in this case $t(\omega(i)) = j_k$. Since $i \in M_{j_k}$, we have that $\omega(i) \in M_{j_k} \setminus \{\omega(j_k)\}$. If $t(\omega(i)) = j_k$ we are done, so assume that $t(\omega(i)) = k$. By condition (i) of Theorem 2 $k \in M_{\omega(i)}$, and by condition (ii) of Theorem 2, since $k \succ_{\omega(i)} j_k$, $\{k\} = M^{\omega(i)}_{j_k}$. Then, $k = i$, with $i \in M_{\omega(i)} \setminus \{\omega(\omega(i))\}$. Hence, by condition (ii) of Theorem 2 again, we have that for any $j \in M_i \setminus \{\omega(i)\}$, $j \succ_i \omega(i)$, then $\{j\} = M^i_{\omega(i)}$. But $|M_i \setminus \{\omega(i)\}| > 1$, and then $j \in M^i_{\omega(i)}$ for all $j \in M_i \setminus \{\omega(i)\}$, contradicting the uniqueness of $M^i_{\omega(i)}$.

P2. Let i be an agent such that $|M_i| \leq 2$. We will prove that either $t(i) \in T$ with $t(t(i)) \in \{i, \omega(i)\}$ or agent i belongs to a ring S such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} r$ for any $r \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Consider first that $M_i = \emptyset$, then we have $i \succsim_i j$ for all $j \in N$, and so $t(i) \in T$ with $t(t(i)) = \{i\}$.

Consider now that $M_i = \{j\}$. If $t(j) = i$ we are done, so assume that $t(j) = k$ with $k \neq i$. By the reasoning in [**P1.**], if $|M_j| > 2$, then $t(k) = j$ and we are done. So let $|M_j| \leq 2$. Since $k \in M_j \setminus \{\omega(j)\}$, by condition (ii) of Theorem 2, either $t(k) = j$ or there exists an agent $l \in M_k$ such that $l \succ_k j$ and $\{l\} = M^k_j$. However, this implies $l = i$ (since $i \in M^k_j$), which contradicts condition (i) of Theorem 2 since this implies that $i \succ_k j$ when by the initial assumption $k \notin M_i$. Hence, if $M_i = \{j\}$, then either $t(j) = i$ or $t(j) = k$ with $t(k) = j$.

Consider now the case that M_i has two elements. Without loss of generality, let $M_i = \{j, k\}$ with $j \succ_i k$. Since $j \in M_i \setminus \{\omega(i)\}$ from condition (ii) of Theorem 2, we deduce that either $t(j) = i$ or $t(j) = k$. Let assume that $t(j) = k$, otherwise we are done. We will show that either $t(k) = j$ or there exists a ring $S = \{i, j, k\}$ such that $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} r$ for all $r \in N \setminus \{s_{i+1}, s_{i-1}\}$.

(Until now, we have assumed that $j \succ_i k$ and $k \succ_j i$.) Assume that there exists an agent $s \in M_j \setminus \{i, k\}$ such that $s \succ_j i$. Since $j \in M_i \setminus \omega(i)$, by condition (ii) of Theorem 2, $\{s\} = M^j_i$, which implies $s = k$. Therefore, there cannot be any agent different from k more preferred than i in agent j's preferences.

Consider now that there exists an agent $s \in M_j \setminus \{i,k\}$ such that $i \succ_j s$. Then $|M_j| > 2$ and by the reasoning of [**P1.**], $t(j) = k$ implies $t(k) = j$ as desired.

Finally, suppose that there is no $s \in M_j \setminus \{i,k\}$. (That is, $M_j = \{k,i\}$ with $k \succ_j i$.) If $t(k) = j$ we are done, so assume that $t(k) = l \neq j$. Since $k \in M_j \setminus \{\omega(j)\}$ and there exists $l \in M_k$ such that $l \succ_k j$, by condition (ii) of Theorem 2 $\{l\} = M_j^k$, which implies that $l = i$ and then $i \succ_k j$. Now we prove that there cannot be any agent between i and j in agent k's preferences. Suppose there is an agent $s \in M_k \setminus \{i,j\}$ such that $s \succ_k j$. Since $k \in M_j \setminus \{\omega(j)\}$ and $s \succ_k j$, then by condition (ii) of Theorem 2 $M_j^k = \{s\}$ and then $s = i$. Therefore, we have that $S = \{i,j,k\}$ form a ring in P such that $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} r$ for any $r \in N \setminus \{s_{i+1}, s_{i-1}\}$ as we wanted to prove. □

Proof. [**Proof of Corollary 1**]

According to Proposition 2, by properties [**P1.**] and [**P2.**], if $i \notin T$, then $|M_i| \leq 2$. If $M_i = \{j\}$, by property [**P2.**] $j \in T$. If $M_i = \{j,k\}$ and $i \notin S$, then by property [**P2.**] $j, k \in T$. □

Proof. [**Proof of Proposition 3**]

We will show that (N,P) satisfies conditions (i) and (ii) of Theorem 2. Condition (i) is trivially satisfied since all agents of N are mutually acceptable between them, that is $\forall i \in N, A_i \setminus M_i = \emptyset$. Now let $N = \{i,k,l\}$ and assume w.l.o.g. that $k \in M_i \setminus \{\omega(i)\}$. If $i \succ_k l$, for agent i condition (ii) is satisfied by default. So let $l \succ_k i$. Since $l = \{\omega(i)\} = M_i^k$, condition (ii) is thus also satisfied. □

Proof. [**Proof of Proposition 4**]

($\Rightarrow$) The existence of a ring S in the preferences with $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$, for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$ is a sufficient condition for non-existence of stable matchings in any one-to-one matching problem (dominance invariant or not). Let μ be a matching such that $\mu(s_i) = j$ for some $s_i \in S$ and some $j \notin S$. This matching is blocked by the pair $\{s_i, s_{i-1}\}$. Therefore any matching containing a pair formed by an agent in the ring and an agent outside the ring is not stable. Consider then a matching μ' satisfying that $\mu'(s_i) = s_{i+1}$ and $\mu'(s_{i-1}) = s_{i-1}$ This matching is blocked by the pair $\{s_{i-1}, s_{i+1}\}$. Therefore any matching in which agents in S are matched among themselves is not stable. Hence, there is no matching stable as we wanted to prove.

($\Leftarrow$) We will show that if a one-to-one matching problem is dominance invariant and unsolvable then there exists a ring S in P satisfying that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.

First, we show that if a problem (N, P) is dominance invariant, there cannot be a ring S in P with $|S| > 3$. By contradiction, suppose there is a ring S with $|S| > 3$ and take agent $s_i \in S$. By definition of ring, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} s_i$ and $s_{i+2} \succ_{s_{i+1}} s_i \succ_{s_{i+1}} s_{i+1}$. Then $s_{i+1} \in M_{s_i} \setminus \{\omega(s_i)\}$ and $s_{i+2} \succ_{s_{i+1}} s_i$. By condition (ii) of Theorem 2 $M_{s_i}^{s_{i+1}} = s_{i+2}$. That implies that $s_{i+2} = s_{i-1}$, which is only possible if $|S| = 3$.

Now we show that for any agent $s_i \in S$ there cannot be an agent $j \notin S$ such that $j \succ_{s_i} s_{i-1}$. By contradiction, suppose there exists an agent $j \notin S$ and $j \succ_{s_i} s_{i-1}$. By condition (i) of Theorem 2, $j \in M_{s_i}$. By definition of ring, $s_i \succ_{s_{i-1}} s_{i+1}$ and then $s_i \in M_{s_{i-1}} \setminus \{\omega(s_{i-1})\}$. By condition (ii) of Theorem 2, $M_{s_{i-1}}^{s_i} = j$, which implies that $j = s_{i+1}$. But this contradicts that $j \notin S$. □

Proof. [**Proof of Proposition 5**]

For all $i \in T$, it is easy to see that $\mu_C(i) = t(i)$. Otherwise μ_C is blocked by the pair $\{i, t(i)\}$ and it is not stable. (This holds for all one-to-one matching problems dominant invariant or not.) Consider now an agent $j \notin T$ such that $\mu_C(j) \neq j$. Then either $t(\mu_C(j)) \neq j$ or $t(j) \neq \mu_C(j)$. W.l.o.g. assume that $t(j) \neq \mu_C(j)$. By condition (i) of Theorem 2, $t(j) \in M_j$. Let $\mu_C(t(j)) = l$. Since matching μ_C is stable, then $l \succ_{t(j)} j$. By condition (ii) of Theorem 2, $M_j^{t(j)} = l$, which implies $\mu_C(j) = l$. But this is not possible, given that agent l cannot be matched in matching μ_C to agent j and agent $t(j)$ at the same time. □

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