

# Optimal Strategies in $n$ -Person Unilaterally Competitive Games\*

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## Abstract

In this paper, we prove that the concept of value traditionally defined in the class of two-person zero-sum games can be adequately generalized to the class of  $n$ -person weakly unilaterally competitive games introduced by Kats & Thisse [KT92b]. We subsequently establish that if there exists an equilibrium in a game belonging to the latter class, then every player possesses at least an optimal strategy (i.e., a strategy yielding at least the value to this player). Furthermore, we show that, in all unilaterally competitive games that have a Nash equilibrium profile, a strategy profile is an equilibrium if and only if it is an optimal profile. From these results, we deduce a very strong foundation to the Nash equilibrium concept in unilaterally competitive games.

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# 1 Introduction

The class of two-person zero-sum games has always played a central role in the theory of non-cooperative games. The important equilibrium idea, of which the notion of Nash equilibrium is an essential generalization, arises very naturally in this class of games. The reason is that the existence of a value of the game and of optimal strategies (strategies that yield at least the value against any choice of the opponent) carries some very interesting and powerful properties.

First, the foundation of the equilibrium solution concept in this class, is very strong<sup>1</sup>. The computation of an equilibrium is not an issue: the game can be solved and the outcome is strictly determined (it is the value of the game). The key feature is that it is possible to compute optimal strategies of one player without computing those of her opponent. The problem of finding equilibrium pairs can thus be split up into two easier problems: the choice of optimal strategies by a given player can be done by assuming that her opponent behaves as a cut-throat automaton whose sole intention is to minimize her payoff. Finally, the existence of optimal strategies implies that all Nash equilibria yield the same payoff vector. This is important in some classes of extensive games with two steps. Indeed, if in the second stage every equilibrium gives the same payoff then, one is able to use backward induction to solve the whole game. Typical examples are Hotelling Location - Price fixing problems (see [KT92a] for instance).

Unfortunately, the elegant theory of two-person zero-sum games is not appropriate when applying game theory to economic or other social sciences situations. Indeed, although such games have been widely studied in game theory, most of the games that bear some interest in the social sciences are non-zero-sum ones. In order to be useful for economic applications, a class of games needs to represent more realistic situations and has to deal with more than two players.

A traditional theme among game theorists is to preserve some of the elegant properties of the two-person zero-sum games within enlarged class of games<sup>2</sup>. Because, the antagonistic nature of two-person zero-sum games is the source of the interesting characteristics, theorists have naturally paid attention to the classes of games that are “com-

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<sup>1</sup>In general, this is not necessarily the case. See [DWF98] and [DW98] for details.

<sup>2</sup>See, among others, [vNM47], [Aum61],[Sha64], [Ros74], [MV76], [Fri83], [KT92b], [DW99], [Bea99].

petitive” in the sense that players do not have many incentives to cooperate nor correlate their plans.

In this article, we study the class of  $n$ -person (weakly) unilaterally competitive games, introduced by Kats & Thisse [KT92b]. We show that this class shares some very nice properties with the class of two-person zero-sum games. Indeed, we establish that, even when the number of players is greater than two, every player can guarantee herself a Nash equilibrium payoff by selecting a Nash equilibrium strategy. From this result, we define an extension of the notion of value and we prove that, if there exists an equilibrium<sup>3</sup>, optimal strategies always exist. We also state that in a unilaterally competitive game possessing an equilibrium, if one selects an optimal strategy for each player then the resulting profile is always an equilibrium and all equilibria can be constructed in this way. This result is important because similarly to the case of the class of two-person zero-sum games, it is possible to compute the optimal strategies of the players independently of each other. We thus prove that in unilaterally competitive games the foundation of Nash equilibria is very strong.

Our results on the existence of a value of the game and of optimal strategies imply Kats & Thisse’s main theorems: for all weakly unilaterally competitive games, each of the players receives the same payoff at all equilibrium points and for all unilaterally competitive games, equilibrium strategies are exchangeable. We give some very simple, short and intuitive proofs of their results.

The paper is organized as follows. Section 2 consists of preliminaries where we recall the important notions of maximin and minimax strategies. In section 3, we define the class of weakly unilaterally competitive games and illustrate it with economic applications. Among other examples, we develop a new model of product differentiation. Section 4 contains the main results of the paper. Section 5 concludes by giving a very peculiar example of a unilaterally competitive game.

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<sup>3</sup>All our results hold (in particular) for pure strategies, so that such an assumption is sensible.

## 2 Preliminaries

A finite noncooperative normal form game<sup>4</sup>  $\Gamma$  is a triple  $\{n, \Sigma, (u_i)_{1 \leq i \leq n}\}$  where  $n$  is the number of players<sup>5</sup>,  $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$  is a compact set and for every  $i$  ( $i = 1, \dots, n$ ),  $u_i$  is a continuous mapping from  $\Sigma$  to  $\mathbb{R}$ .  $\Sigma$  represents the set of strategy profiles. Each player  $i$  simultaneously chooses a strategy  $s_i$  in  $\Sigma_i$ . Notice that  $\Sigma_i$  can be interpreted as a set of either pure or mixed strategies. Player  $i$ 's payoff is given by her utility function  $u_i$ . We frequently denote all players, other than some given player  $i$ , as  $-i$ .

**Definition 2.1** *Let  $\Gamma = \{n, \Sigma, u\}$  be a game.*

- *The maximin value or the gain-floor for player  $i$  is the maximum payoff that she can guarantee to herself; it is her best security level:*

$$\underline{v}_i := \max_{s_i \in \Sigma_i} \min_{s_{-i} \in \Sigma_{-i}} u_i[s_i, s_{-i}]$$

- *A maximin strategy for player  $i$ ,  $s_i$ , is a strategy that maximizes her security level. Formally, for all strategy profiles  $s_{-i}$  of players  $-i$ , we have that  $u_i[s_i, s_{-i}] \geq \underline{v}_i$ . Whatever the other players play, by playing a maximin strategy, player  $i$  cannot receive less than her maximin payoff.*
- *The minimax value or loss-ceiling for player  $i$  is the lowest payoff that the other players can force upon player  $i$ :*

$$\bar{v}_i := \min_{s_{-i} \in \Sigma_{-i}} \max_{s_i \in \Sigma_i} u_i[s_i, s_{-i}]$$

- *A minimax strategy,  $s_{-i}$ , is a strategy such that, for all strategies  $s_i$  of player  $i$ ,  $u_i[s_i, s_{-i}] \leq \bar{v}_i$ . Whatever player  $i$  does, if the other players choose a minimax strategy, then she cannot receive more than her minimax value.*

Maximin and minimax strategies always exist under our assumptions, but need not be unique. If the maximin and the minimax values are equal for each player, we name this  $n$ -uple the value of the game. In that case, by choosing a maximin strategy, a player always receives at least her value against any choice of her opponents. In this context, because they always yield the best possible outcome, we define maximin strategies as optimal.

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<sup>4</sup>Or simply a game.

<sup>5</sup>We always assume that the number of players is at least two.

Note that it is easy to prove that in any game, a player's equilibrium payoff is always greater than the loss-ceiling of that player which, in turn, always exceeds (weakly) her gain-floor: if  $s^*$  is an equilibrium,  $\underline{v}_i \leq \bar{v}_i \leq u_i[s^*]$ . Lemma 2.1, whose proof is trivial, gives a set of sufficient conditions for a profile to be a Nash equilibrium.

**Lemma 2.1** *Let  $\Gamma = \{n, \Sigma, u\}$  be a game. If for every player  $i$ , the profile  $s_{-i}^*$  is minimax and the payoff  $u_i[s^*]$  is equal to  $\bar{v}_i$ , then  $s^*$  is a Nash equilibrium of  $\Gamma$ .*

## 3 (Weakly) Unilaterally Competitive Games

### 3.1 Definitions

In their paper, Kats & Thisse define a new class of games: the weakly unilaterally competitive games (see [KT92b]). To quote them,

A game belongs to this class if a unilateral move by one player which results in an increase in that player's payoff also causes a (weak) decline in the payoffs of all other players. Furthermore, if that move causes no change in the mover's payoff then all other players' payoffs remain unchanged.

**Definition 3.1** *A game  $\Gamma = \{n, \Sigma, u\}$  is weakly unilaterally competitive if:*

$$\begin{aligned} &\forall i, \forall s_i, s'_i \in \Sigma_i, \forall s_{-i} \in \Sigma_{-i} \\ &u_i[s_i, s_{-i}] > u_i[s'_i, s_{-i}] \implies u_{-i}[s_i, s_{-i}] \leq u_{-i}[s'_i, s_{-i}] \quad (1) \\ &\text{and } u_i[s_i, s_{-i}] = u_i[s'_i, s_{-i}] \implies u_{-i}[s_i, s_{-i}] = u_{-i}[s'_i, s_{-i}] \end{aligned}$$

The two authors slightly strengthen their definition and define a game as unilaterally competitive if any unilateral change of strategy by one player results in a (weak) increase (resp. decrease) in that player's payoff if and only if this change in strategy results in a (weak) decline (resp. increase) in the payoffs of all other players.

**Definition 3.2** *A game  $\Gamma = \{n, \Sigma, u\}$  is unilaterally competitive if:*

$$\begin{aligned} &\forall i, \forall s_i, s'_i \in \Sigma_i, \forall s_{-i} \in \Sigma_{-i} \\ &u_i[s_i, s_{-i}] \geq u_i[s'_i, s_{-i}] \iff u_{-i}[s_i, s_{-i}] \leq u_{-i}[s'_i, s_{-i}] \quad (2) \end{aligned}$$

## 3.2 Some Illustrative Economic Examples

We now present three economic models formalized as weakly unilaterally competitive game<sup>6</sup>. We first expose the standard model of private provision of a public good and we follow by giving an example of oligopolistic competition inspired by the Bertrand model. The third example is an alternative model of product differentiation.

### 3.2.1 Private Provision of a Public Good

Consider a desirable<sup>7</sup> public good that is produced by means of private contribution by  $n$  consumers  $i = 1, \dots, n$ . We will focus on the case in which exclusion is not possible: no customer can be excluded from the benefits of the public good. The larger amount of public good provided, the more each consumer benefits, but each would prefer the other customers to incur the cost of supplying the good. Consumers decide simultaneously how much to contribute to the public good ; let  $x_i \in \mathbb{R}^+$  be the amount of the public good produced by consumer  $i$ . The total amount of the public good provided by customers is then  $x = \sum_{j=1}^n x_j$ . We denote consumer  $i$ 's utility from the public good by  $\Phi_i(x)$ . The cost of supplying  $q$  units of the public good is  $c(q)$ . Customer  $i$ 's total utility function  $\Psi_i$  is the difference between her utility provided by the aggregate level of production and her cost of production:  $\Psi_i(x_1, \dots, x_n) = \Phi_i(x) - c(x_i)$ .

We assume that the benefits provided to player  $i$  by increasing her own purchasing share does not compensate the increase of the cost. In other words, each player benefits when the public good is provided, but each would prefer the others to incur the cost of supplying it<sup>8</sup>. Formally, we assume that  $0 < \Phi'_i(\sum_{j=1}^n x_j) < c'(x_i)$ . Because of this assumption, by unilaterally increasing (resp. decreasing) its contribution, player  $i$ 's utility increases (resp. decreases) while that of her opponents decreases (resp. increases). This game is thus weakly unilaterally competitive.

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<sup>6</sup>It seems that the class of unilaterally competitive games is less attractive from the point of view of applied game theory.

<sup>7</sup>Note that in what follows, the public good need not necessarily be desirable (e.g., pollution problem).

<sup>8</sup>This is the key assumption of the *free-rider problem*.

### 3.2.2 A Model of Price Competition

It is widely observed that the price of retail goods is very rarely a round amount<sup>9</sup>. More generally, product prices end with 99, 95 and even sometimes 89. We present a model in which a continuum of customers want to buy a unit (per consumer) of a homogeneous commodity. Without loss of generality, we normalize the measure of the set of consumers to 1. The commodity is sold by three producers. Each producer has to select a price at which she wants to sell a unit of commodity. Only three different prices are available: producers have to choose a price belonging to the set  $\{89, 95, 99\}$ . The consumers buy the goods to the producer that sell it at the minimum price. If several producers propose the commodity for the same price, we assume that each one has an equal probability to sell it to the consumers. The payoffs for each producer are given in Figure 1.

	89	95	99
89	29.7, 29.7, 29.7	44.5, 0, 44.5	44.5, 0, 44.5
95	0, 44.5, 44.5	0, 0, 89	0, 0, 89
99	0, 44.5, 44.5	0, 0, 89	0, 0, 89

	89	95	99
89	44.5, 44.5, 0	89, 0, 0	89, 0, 0
95	0, 89, 0	31.7, 31.7, 31.7	47.5, 0, 47.5
99	0, 89, 0	0, 47.5, 47.5	0, 0, 95

	89	95	99
89	44.5, 44.5, 0	89, 0, 0	89, 0, 0
95	0, 89, 0	47.5, 47.5, 0	95, 0, 0
99	0, 89, 0	0, 95, 0	33, 33, 33

Figure 1: A model of price competition

It is easy to verify that this game is weakly unilaterally competitive. Also note that for each player  $i$ , the strategy 99 is dominated. Indeed, both strategies 89 and 95 yield a better payoff to player  $i$  against any players  $-i$ 's strategy profile. If strategy 99 is eliminated, then the strategy 95 is dominated for all players. The equilibrium strategy, 89

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<sup>9</sup>A rationale for this phenomenon has been given in the marketing literature (e.g. [Wil90]).

is not dominated.

### 3.2.3 Location under Nonprice Competition

Product positioning, be it in geographical space or within the space of product attributes, is a major concern for the firms as soon as consumers are heterogeneous. This is especially true when price is not under the control of the firms (which may happen for a variety of reasons including cartel agreements or regulation). In this case indeed, firms battle for market shares through location choices and the game is likely to be highly competitive. Should a firm specialize on a limited market “niche” or sell a more basic product that competes with all others? We provide hereafter a framework that formalizes this problem by allowing firms either to concentrate on their core market or to steal other firm market shares through location choices.

Consider  $n$  lines of length one that have one common endpoint called the “center”.  $n$  sellers  $1, \dots, n$  of a homogeneous product with zero production cost are installed at respective distances  $x_1, \dots, x_n \in [0, 1]$  from the center (see Figure 2). Each seller owns one and only one line and can choose any location on this line. Customers are distributed uniformly along the lines and each one consumes exactly one unit of the commodity. We assume that the mill price is given and equal for all firms. Since the product is homogeneous, the price is fixed and, assuming that consumers pay for the transportation cost, a customer will purchase the good to the nearest firm. When several firms are equidistant from a consumer, we assume that each one has an equal probability to sell to the customer.

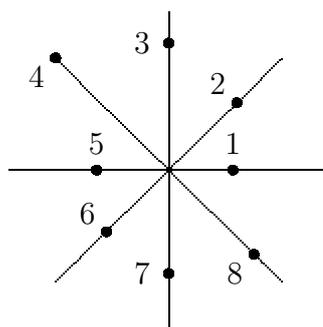


Figure 2: Particular location of 8 firms

The strategies of the firms are given by locations only. Because the firm’s payoff only depends on the level of sales, it is given by the measure of the set of consumers they serve. Consider the set  $M$  of

firms that are located the nearest from the center:  $M = \{i : x_i = \min(x_1, \dots, x_n)\}$

$$u_i(x) = \begin{cases} 1 + \frac{1}{\#M} \sum_{j \notin M} \left( \frac{\text{dist}[x_i, x_j]}{2} - x_i \right) & \text{if } x_i \in M \\ = 1 - \frac{n - \#M}{2\#M} x_i + \frac{\sum_{j \notin M} x_j}{2\#M} \\ 1 - x_i + \frac{\text{dist}[x_i, \min(x_{-i})]}{2} & \text{if } x_i \notin M \\ = 1 - \frac{x_i}{2} + \frac{\min(x_{-i})}{2} \end{cases}$$

Note that if seller  $i$  does not belong to  $M$ , her utility,  $u_i(x)$ , is always lower than 1: she does not serve her whole line. On the contrary, if she belongs to  $M$  then her payoff always exceeds 1 because she serves at least her entire line.

Now, let us assume that seller  $i$  decides to unilaterally change her location from  $\hat{x}_i$  to  $x_i$ . Without loss of generality, consider that  $x_i > \hat{x}_i$ . There are three different cases:

- $x_i, \hat{x}_i \notin M$ . In that case, seller  $i$ 's utility increases of  $\frac{x_i - \hat{x}_i}{2}$ , the payoff of every seller belonging to  $M$  decreases of an amount  $\frac{x_i - \hat{x}_i}{2\#M}$  and all the other sellers' payoffs stay unchanged.
- $x_i \notin M$  and  $\hat{x}_i \in M$ . In that case, seller  $i$ 's payoff increases because her payoff was below 1 in  $x_i$  and becomes greater than 1 in  $\hat{x}_i$ . Every seller belonging to  $M$  decrease while all the other sellers' payoffs stay unchanged.
- $x_i, \hat{x}_i \in M$ . Let  $m$  be the number of sellers in  $M$ . When seller  $i$  is in  $x_i$ , she increases her payoff of an amount  $\frac{(m-1)x_j - m\hat{x}_i + x_i}{2m}$  while all the other sellers' payoffs decrease.

This model of nonprice competition is thus a weakly unilaterally competitive game.

### 3.3 Value and Optimal Strategies

We first state that in a weakly unilaterally competitive game, playing an equilibrium strategy  $s_i^*$  guarantees player  $i$  her equilibrium payoff  $u_i[s^*]$  independently of the strategy chosen by her opponents.

**Theorem 3.1** *Let  $\Gamma = \{n, \Sigma, u\}$  be a weakly unilaterally competitive game.*

$$s^* \text{ equilibrium of } \Gamma \implies \forall i \quad \min_{s_{-i} \in \Sigma_{-i}} u_i[s_i^*, s_{-i}] = u_i[s^*] \quad (3)$$

**Proof** We prove the result by induction on the number of players. We first show that statement (3) always holds in a two-person weakly unilaterally competitive game. On the contrary, assume that  $\exists s_{-i} \in \Sigma_{-i} \quad u_i[s_i^*, s_{-i}] < u_i[s^*]$ . Because the game  $\Gamma$  is a weakly unilaterally competitive game, we know that  $u_{-i}[s_i^*, s_{-i}] > u_{-i}[s^*]$ . But this contradicts the fact that  $s^*$  is an equilibrium.

Second, we prove that if the result is correct in the  $(n - 1)$  players case, it has to remain valid in the  $n$  players case. Note that if the following statement (4) is correct the result is proved

$$\forall s_{-i} \in \Sigma_{-i} \exists j \neq i \quad u_j[s_i^*, s_j^*, s_{-ij}] \geq u_j[s_i^*, s_{-i}] \quad (4)$$

Indeed, by applying the weakly unilaterally competitive property, assertion (4) becomes  $u_i[s_i^*, s_j^*, s_{-ij}] \leq u_i[s_i^*, s_{-i}]$ . By fixing  $s_j^*$  we know by the inductive hypothesis that  $u_i[s_i^*, s_j^*, s_{-ij}] \geq u_i[s^*]$ . Putting together the last two inequalities, we obtain the desired result.

By contradiction, assume that statement (4) is false:

$$\exists s_{-i} \in \Sigma_{-i} \quad \forall j \neq i \quad u_j[s_i^*, s_j^*, s_{-ij}] < u_j[s_i^*, s_{-i}] \quad (5)$$

This implies that the profile  $s_{-i}$  is a strict Nash equilibrium of the restricted game  $\Gamma_{-i}(s_i^*) = \{n - 1, \hat{\Sigma}, u[s_i^*, .]\}$  where the players are those of  $\Gamma$  except player  $i$  and the strategy space is restricted to  $\hat{\Sigma} = \prod_{j \neq i} \{s_j^*, s_j\}$ . Thus, by the inductive hypothesis applied on  $\Gamma_{-i}(s_i^*)$  which is a weakly unilaterally competitive game, we must have that  $\forall j \neq i \quad u_j[s_i^*, s_{-i}] \leq u_j[s_i^*, s_j^*, s_{-ij}^*]$ . Taking into account this last inequality, the fact that  $s^*$  is a Nash equilibrium of  $\Gamma$  and statement (5) we have that  $\forall j \neq i \quad u_j[s_i^*, s_j^*, s_{-ij}] < u_j[s^*]$ . But this contradicts the inductive hypothesis on every player  $j$  (by fixing  $s_i^*$ ).  $\square$

Note that the proof of Theorem 3.1 does not use the fact that  $s_i^*$  is an equilibrium strategy. Indeed, consider any strategy  $s_i^*$  of player  $i$ . The proof establishes that if  $s_{-i}^*$  is a Nash equilibrium of the  $(n - 1)$ -person game obtained when strategy  $s_i^*$  is fixed then  $s_i^*$  is a maximin strategy of the original game.

As a corollary, we prove that for all weakly unilaterally competitive games that have a Nash equilibrium profile, the maximin value, the minimax value and all the equilibrium payoffs of any player are equal. We denote this number as the value of the game for that player. It is also shown that any equilibrium strategy is a maximin strategy and

that by selecting Nash equilibrium strategies, player  $-i$  guarantee that player  $i$  gets at most her value.

**Corollary 3.1** *Let  $\Gamma = \{n, \Sigma, u\}$  be a weakly unilaterally competitive game. If  $s^*$  is a Nash equilibrium then  $u_i[s^*] = \underline{v}_i = \bar{v}_i$ . Furthermore, for every player  $i$ ,  $s_i^*$  and  $s_{-i}^*$  are always maximin and minimax strategies respectively.*

**Proof** By Theorem 3.1, we know that  $\underline{v}_i \geq u_i[s^*]$ . But because  $s^*$  is a Nash equilibrium we also know that  $\underline{v}_i \leq u_i[s^*]$ . Thus we have that  $u_i[s^*] = \underline{v}_i = \bar{v}_i$ . By replacing  $\underline{v}_i$  and  $\bar{v}_i$  by  $u_i[s^*]$  in the definition of maximin and minimax strategies the remaining of the proof is straightforward (using Theorem 3.1 and the fact that  $s^*$  is a Nash equilibrium).  $\square$

Corollary 3.1 is very powerful because it proves the existence of a value of the game which in turn implies that maximin strategies are optimal (as explained in section 2). Also note that because the value is defined independently of the equilibria, this result trivially implies Kats & Thisse's main theorem: In all weakly unilaterally competitive games, all equilibrium payoffs are equal. Our proof, based on Theorem 3.1, is simpler, shorter and more intuitive than Kats & Thisse's one.

Theorem 3.1 and Corollary 3.1 assert that a strategy needs to be optimal in order to be an equilibrium one. We now prove that for all unilaterally competitive games that have an equilibrium, to be an optimal strategy is also a sufficient condition. More precisely, Theorem 3.2 proves that a strategy is maximin if and only if it is a Nash equilibrium strategy.

**Theorem 3.2** *Let  $\Gamma = \{n, \Sigma, u\}$  be a unilaterally competitive game. If there exists a Nash equilibrium, say  $s^*$ , then the profile  $(\underline{s}_i, s_{-i}^*)$  is an equilibrium if and only if  $\underline{s}_i$  is a maximin strategy.*

**Proof** Note that corollary 3.1 implies directly the only if part of the proof. To prove the other part, consider any maximin strategy,  $\underline{s}_i$ , of player  $i$ . We thus know that  $u_i[s^*] \leq u_i[\underline{s}_i, s_{-i}^*]$ . From the fact that  $s^*$  is an equilibrium we deduce that player  $i$  has no incentive to unilaterally deviate from  $(\underline{s}_i, s_{-i}^*)$  and also that  $u_i[s^*] = u_i[\underline{s}_i, s_{-i}^*]$  which in turn implies by the (weakly) unilaterally competitive property of  $\Gamma$  that

$$\forall l \quad u_l[s^*] = u_l[\underline{s}_i, s_{-i}^*] \quad (6)$$

There remains to prove that no player  $j \neq i$  has an incentive to unilaterally deviate from the profile  $(\underline{s}_i, s_{-i}^*)$ . We give a different proof of this assertion for the two player case and the (strictly) more than two players case.

In order to prove the result in the two-person case, note that because  $\underline{s}_i$  is a maximin strategy for player  $i$ , we know that  $\forall s_j \in \Sigma_j u_i[s^*] \leq u_i[\underline{s}_i, s_j]$ . This implies, together with the fact that  $u_i[s^*] = u_i[\underline{s}_i, s_{-i}^*]$ , that  $\forall s_j \in \Sigma_j u_i[\underline{s}_i, s_{-i}^*] \leq u_i[\underline{s}_i, s_j]$ . Applying the unilaterally competitive property of  $\Gamma$ , we obtain the desired statement.

To show the result in the strictly more than two players case, let  $s_j$  be an arbitrary strategy of player  $j$  and let us consider a third player  $k \notin \{i, j\}$ . From Theorem 3.1, we deduce the following inequality:  $u_k[s^*] \leq u_k[\underline{s}_i, s_j, s_{-ij}^*]$ . This, in turn, implies, by statement (6), that  $u_k[\underline{s}_i, s_{-i}^*] \leq u_k[\underline{s}_i, s_j, s_{-ij}^*]$ . This inequality holds for every  $k \notin \{i, j\}$  as well as for  $k = i$  (by the above reasoning). Hence, by the unilaterally competitive property of  $\Gamma$ , player  $j$  has no incentive to unilaterally deviate from the profile  $(\underline{s}_i, s_{-i}^*)$  by playing  $s_j$ .  $\square$

The weakly unilaterally competitive property of  $\Gamma$  is not enough to prove Theorem 3.2, as the example in Figure 3 demonstrates. Indeed, in this weakly unilaterally competitive game, the strategy profile  $(u, l)$  is an equilibrium and player 1's strategy  $d$  is a maximin strategy while the profile  $(d, l)$  is not an equilibrium.

	$l$	$r$
$u$	0, 1	1, 0
$d$	0, 1	0, 2

Figure 3: A WUC game which does not satisfy the result of Theorem 3.2

From Corollary 3.1, we know that if  $s^*$  is a Nash equilibrium then  $s_{-i}^*$  is a minimax strategy of players  $-i$ . The example presented in Figure 4 demonstrates that the opposite assertion does not hold for all unilaterally competitive games possessing an equilibrium: a minimax strategy need not be part of a Nash equilibrium. Indeed, in this game,  $(u, l, A)$  is the unique Nash equilibrium while  $(d, r)$  is a minimax strategy of players 1 and 2.

	$l$	$r$
$u$	0, 0, 0	1, -1, 0
$d$	-1, 1, 0	0, 0, 0
	$A$	

	$l$	$r$
$u$	1, 1, -1	2, 0, -1
$d$	0, 2, -1	1, 1, -1
	$B$	

Figure 4: A UC game with a minimax strategy that is not a Nash equilibrium strategy

Nevertheless, Theorem 3.3 asserts that in all two-person unilaterally competitive games, the set of minimax strategies, the set of maximin strategies and the set of equilibrium strategies always coincide for every player.

**Theorem 3.3** *Let  $\Gamma = \{2, \Sigma, u\}$  be a two-person unilaterally competitive game which has an equilibrium pair. Consider a player  $i$ 's strategy  $s_i \in \Sigma_i$ . The three following statements are equivalent: (a)  $s_i$  is maximin; (b)  $s_i$  is minimax; (c)  $s_i$  is a Nash equilibrium strategy.*

**Proof** In [LR57], Luce & Raiffa prove this result for the set of all two-person strictly competitive games<sup>10</sup> which is a proper subset of the class of two-person unilaterally competitive games. Their proof trivially holds also for the latter class.  $\square$

We know from Theorem 3.1 and 3.2 that in a unilaterally competitive game, a strategy of a player is optimal if and only if it is an equilibrium strategy for that player. Theorem 3.4 establishes a stronger result: a strategy profile is an equilibrium if and only if it is a profile of optimal strategies.

**Theorem 3.4** *Let  $\Gamma = \{n, \Sigma, u\}$  be a unilaterally competitive game. If there exists an equilibrium,*

$$s \text{ equilibrium of } \Gamma \iff \forall i \quad \underline{s}_i \text{ is a maximin strategy} \quad (7)$$

**Proof** By Corollary 3.1, we know that every equilibrium strategy is a maximin strategy. Assume now that there exists an equilibrium  $s^*$  and denote as  $\underline{s}$  a profile of maximin strategies. Because  $\underline{s}_1$  is a maximin strategy and  $s^*$  is an equilibrium profile, Theorem 3.2 implies that the profile  $(\underline{s}_1, s^*_{-1})$  is a Nash equilibrium. But then because  $\underline{s}_{-2}$  is maximin, we also have that  $(\underline{s}_{12}, s^*_{-12})$  is an equilibrium profile. Iterating over the players, we obtain the result.  $\square$

<sup>10</sup>A two-person game is strictly competitive if all possible outcomes are Pareto optimal (see, for example, [Fri83]).

Theorem 3.4 is important because it implies that, in order to find a Nash equilibrium, one only has to compute maximin strategies. This justifies the terminology we used, describing maximin strategies as optimal ones: if players  $-i$  all play a maximin strategy  $\underline{s}_i$ , then  $\underline{s}_{-i}$  is minimax against player  $i$ , by Theorem 3.4 and Corollary 3.1. Similarly to the two-person zero-sum game case,  $n$ -person unilaterally competitive games can be solved and their outcome are strictly determined.

Also notice that Theorem 3.4 implies directly another Kats & Thisse result: in all  $n$ -person unilaterally competitive games, Nash equilibria are interchangeable<sup>11</sup>.

So far, all our results concerning the class of (weakly) unilaterally competitive games are interesting only if there exists a Nash equilibrium profile. Theorem 3.5 asserts that for all two-person unilaterally competitive games, it is very easy to check the existence of Nash equilibrium. Indeed, this theorem proves that in this class, the maximin and the minimax values are equal for all players if and only if there exists an equilibrium.

**Theorem 3.5** *Let  $\Gamma = \{2, \Sigma, u\}$  be a two-person unilaterally competitive game.  $\Gamma$  possesses an equilibrium if and only if for both players  $\bar{v}_i = \underline{v}_i$ .*

**Proof** Denote respectively by  $\underline{s}_i$  and  $\bar{s}_i$  a maximin and a minimax strategy of player  $i$ , and let  $v_i$  be player  $i$ 's value. From the fact that  $\underline{s}_i$  is maximin and  $\bar{s}_{-i}$  is minimax, we deduce that  $u_i[\underline{s}_i, \bar{s}_{-i}] = v_i$  and thus, we also have that  $u_i[\underline{s}_i, \underline{s}_{-i}] \geq u_i[\underline{s}_i, \bar{s}_{-i}] \geq u_i[\bar{s}_i, \bar{s}_{-i}]$ . Applying the unilaterally competitive property of  $\Gamma$  to both inequalities of the latter assertion, we obtain that  $u_{-i}[\underline{s}_i, \underline{s}_{-i}] \leq u_{-i}[\underline{s}_i, \bar{s}_{-i}] \leq u_{-i}[\bar{s}_i, \bar{s}_{-i}]$ . Consequently, for all players  $i = 1, 2$ , we have that  $u_i[\underline{s}] = u_i[\bar{s}] = v_i$ . This implies that both profiles  $\underline{s}$  and  $\bar{s}$  are Nash equilibria.  $\square$

The example presented in Figure 5 is a weakly unilaterally competitive that does not have any equilibrium while possessing a value. The result of Theorem 3.5 is thus not valid for all weakly unilaterally competitive games.

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<sup>11</sup>Kats & Thisse also prove this result for all two-person weakly unilaterally competitive games.

	$l$	$r$
$u$	0, 1	2, 0
$d$	1, 1	1, 2

Figure 5: A WUC game which does not satisfy the result of Theorem 3.5

## 4 Concluding Remarks

The basic intuition in this paper is that if an  $n$ -person game is competitive enough (and has at least an equilibrium) then the best choice a player can make is to select a cautious strategy (that is a maximin strategy). Furthermore, the resulting recommended profile is always a Nash equilibrium. Our results are powerful in the sense that they predict unambiguously which strategy the players have to play in a competitive game. Therefore, one can search for equilibrium strategies by first finding maximin strategies. Nevertheless, unless it is shown that the game has an equilibrium profile, it is necessary to check that the maximin strategies are equilibrium strategies. Furthermore, remind that Kats & Thisse's contribution and ours are only valuable when there exists a Nash equilibrium. Indeed, if this condition is not satisfied, then the game need not possess a value nor optimal strategies and our results cannot be proved.

None of our results characterizes the (weakly) unilaterally competitive property. Indeed, Figure 6 demonstrates that there exist some non (weakly) unilaterally competitive games that satisfy the results of Theorem 3.1, 3.2, 3.4 and Corollary 3.1 (when strategies are interpreted as mixed)<sup>12</sup>. In fact, there is a unique Nash equilibrium (the profile  $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$  for both players) whose strategies are all maximin (they all yield at least 0 to all players).

	$A$	$B$	$C$
$a$	0, 0	1, 2	2, 1
$b$	2, 1	0, 0	1, 2
$c$	1, 2	2, 1	0, 0

Figure 6: A 2 -person non WUC game satisfying Theorem 3.4.

Also, notice that in all the examples given in this article and in Kats & Thisse's one, at least one player possesses a dominated strategy. A natural question is thus to know whether there exist some uni-

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<sup>12</sup>We borrowed this example from [MV76].

laterally competitive games that are not degenerate in the latter sense. We will differently answer to this question depending on whether the strategies are considered as either pure or mixed. When players only select pure strategies, there exist unilaterally competitive games for which no player has a dominated strategy and which possess a Nash equilibrium, as revealed by Figure 7<sup>13</sup>.

	$l$	$r$	
$u$	2, 2, 2	7, 1, 3	
$d$	1, 3, 7	0, 4, 6	
	$A$		

	$l$	$r$
$u$	3, 7, 1	6, 0, 4
$d$	4, 6, 0	5, 5, 5
	$B$	

Figure 7: A non-degenerate three-person UC game.

Assume now that the definition of (weakly) unilaterally competitive games is understood in the mixed strategy sense, that is, when the strategy set is interpreted as a product of probability distributions. Formally, this means that in Definition 3.1 we have that  $\Sigma_i := \Delta(S_i)$  and  $\Sigma_{-i} := \times_{j \neq i} \Delta(S_j)$ , where for any finite set  $X$ ,  $\Delta(X)$  denotes the set of all probability distributions over  $X$ . The product sets  $S$  and  $\Sigma$  are the pure and mixed strategy set, respectively. The game of Figure 7 which satisfies Definition 3.1 in terms of pure strategies does not satisfy its extension to mixed strategies. Indeed, player 1 is indifferent between the two profiles  $(u, [\frac{1}{2}, \frac{1}{2}], B)$  and  $(d, [\frac{1}{2}, \frac{1}{2}], B)$  while player 2 prefers the latter (yielding  $\frac{11}{2}$  instead of  $\frac{7}{2}$ ). More generally, we show in [DW99] that the unilaterally competitive property is very restrictive when randomized strategies are permitted. We prove that in an  $n$ -person unilaterally competitive game (with  $n$  greater than 2), if  $n - 1$  players have exactly two pure strategies, then there exists a dominated pure strategy for at least one player.

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<sup>13</sup>In fact, similar examples can be built for any number of players and for any number of pure strategies.

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