A bargaining set for roommate problems*

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Abstract

Since stable matchings may not exist, we adopt a weaker notion of stability for solving the roommate problem: the bargaining set. Klijn and Massó (2003) show that the bargaining set coincides with the set of weakly stable and weakly efficient matchings in the marriage problem. First, we show that a weakly stable matching always exists in the roommate problem. However, weak stability is not sufficient for a matching to be in the bargaining set. Second, we prove that the bargaining set is always non-empty. Finally, as Klijn and Massó (2003) get for the marriage problem, we show that the bargaining set coincides with the set of weakly stable and weakly efficient matchings in the roommate problem.

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1 Introduction

Gale and Shapley (1962) introduce a two-sided matching model to answer questions such as who marries whom, who gets which school seat, who shares a dormitory with whom. In their seminal paper, they first introduce the marriage problem, in which there are two disjoint sets of agents, say men and women, and each agent has preferences over agents on the other side of the problem with the possibility of remaining single. No agent can be matched with an agent from the same side. Following the marriage problem, they investigate a generalization, the so-called roommate problem. In the roommate problem, there exists a set of agents each endowed with preferences over all agents. Each agent is interested in forming at most one partnership.[1] A matching is said to be stable if there is no agent who prefers being unmatched to her prescribed partner or no pair of agents prefer being matched to each other to their current partners. They show that stable matchings always exist for the marriage problem, whereas their existence is not guaranteed for the roommate problem. This is a reason why the literature often restricts the analysis to solvable roommate problems (i.e. roommate problems with stable matchings) and on conditions to guarantee the existence of stable matchings (see e.g. Tan, 1991; Chung, 2000; Diamantoudi, Miyagawa and Xue, 2004; Klaus and Klijn, 2010).

In this paper, instead of restricting the analysis to solvable roommate problems, we adopt a weaker notion of stability for solving the roommate problem: the bargaining set. For the marriage problem, Klijn and Massó (2003) adapt a variation of the bargaining set introduced by Zhou (1994) to the marriage problem. They show that the bargaining set coincides with the set of weakly stable and weakly efficient matchings.[2] A matching is weakly stable if all blocking pairs are weak. A blocking pair is said to be a weak blocking pair if a partner of the blocking pair can form another blocking pair with a more preferred partner. In the marriage problem, the existence of weakly stable matchings is guaranteed, since by definition, all stable matchings satisfy weak stability. However, for the roommate problem, the existence of a weakly stable matching does not follow from the existence of a stable matching since such matching may fail to exist.

Our main results follow. First, we guarantee the existence of weakly stable matchings by constructing such a matching even for unsolvable roommate problems. Second, we show that weak stability is not sufficient for a matching to be in the bargaining set. Moreover, when the core is non-empty, in general, it is a strict subset of the set of weakly stable matchings. Third, we prove that the bargaining set is always non-empty. Finally, as Klijn and Massó (2003) do for the marriage problem, we show that the bargaining set coincides with the set of weakly stable and weakly efficient matchings in the roommate

[1] The roommate problem is a model with important applications or extensions including coalition formation (Bogomolnaia and Jackson, 2002), network formation (Jackson and Watts, 2002), kidney exchange problem (Roth, Sönmez and Ünver, 2005) among others. Roth and Sotoymayor (1990) and Manlove (2013) provide a complete survey on matching theory.

[2] A matching is weakly efficient if there does not exist another matching in which all agents are better off.

problem.[3]

The rest of the paper is organized as follows. In Section 2 we introduce the roommate problem and the notion of stability. In Section 3 we extend the notion of weak stability to the roommate problem and we study its structure. In Section 4 we introduce the bargaining set of Zhou (1994) and we investigate its relationship with the set of weakly stable matchings. In Section 5 we conclude.

2 Roommate problems

A *roommate problem* $(N, \succ)$ consists of a finite set of agents N and a preference profile $\succ = (\succ_l)_{l \in N}$. Each player $l \in N$ has a complete and transitive preference ordering $\succ_l$ over N. Throughout the paper, we assume that the preferences are strict. We write that $j \succ_i k$ if agent i strictly prefers j to k, $j \sim_i k$ if i is indifferent between j and k. Since we will consider situations $j = k$, although preferences are assumed to be strict, we need the notation $j \succeq_i k$ meaning that i prefers j at least as well as k, $j \succ_i k$ or $j \sim_i k$. An agent j is *acceptable* to another agent i if $j \succ_i i$. A pair of agents $i, j \in N$ are *mutually best* if i and j are their respective top choices among their acceptable partners.

A *matching* is a one-to-one function $\mu : N \to N$ such that if $\mu(i) = j$, then $\mu(j) = i$. If $\mu(i) = j$, then agents i and j are matched to one another. $\mu(i) = i$ means that agent i is *single* or *unmatched*. Given a roommate problem $(N, \succ)$, we denote the set of all possible matchings by $\mathcal{M}(N, \succ)$. A matching μ is *individually rational* if no agent is matched with an unacceptable partner, that is, $\mu(i) \succeq_i i$ for all $i \in N$. For a given matching μ, a pair of agents $\{i, j\}$ forms a *blocking pair* if they prefer being matched to each other than to their current partners under matching μ, that is, $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. A matching μ is *stable* if it is individually rational and there are no blocking pairs. Gale and Shapley (1962) show that stable matchings may not exist in the roommate problem. A roommate problem is called *solvable* if the set of stable matchings is non-empty, and is called *unsolvable* otherwise. It is well-known that, in the roommate problem, whenever there exist stable matchings, it coincides with the *core*, in which no subset of agents have incentives to be matched among themselves, possibly by dissolving their current partnerships to obtain a strictly better partner.[4]

Definition 1. Given a roommate problem $(N, \succ)$, a *ring* $\mathcal{A} = \{a_1, \ldots, a_k\} \subseteq N$ is an ordered subset of agents, $k \geq 3$, such that (subscript modulo k)

$$a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i \text{ for all } i \in \{1, \ldots, k\}.$$

A ring $\mathcal{A}$ is an *odd ring* if the number of agents in the ring, $|\mathcal{A}|$, is odd.

[3] Other concepts based on a relaxation of the stability notion have been proposed for matching problems. See e.g. Abraham, Biró and Manlove (2006), Mauleon, Vannetelbosch and Vergote (2011), Iñarra, Larrea and Molis (2013), Biró, Iñarra and Molis (2016).

[4] When no confusion arises, we simply denote any coalition by its agents, e.g. ij instead of $\{i, j\} = S \subseteq N$.

We denote the set of all odd rings by $\mathcal{R}$. Tan (1991) and Chung (2000) show that, in any given roommate problem $(N, \succ)$, if the preference profile has no odd rings, there exist stable roommate matchings.

A matching is said to be *weakly efficient* if there is no other matching in which all agents are strictly better off.

Definition 2. Given a roommate problem $(N, \succ)$, a matching $\mu \in \mathcal{M}(N, \succ)$ is *weakly efficient* if there is no matching $\mu' \in \mathcal{M}(N, \succ)$ such that all agents are strictly better off, i.e., $\mu'(i) \succ_i \mu(i)$ for all $i \in N$.

Since stable roommate matchings might not exist, we study the existence of weakly stable matchings in the next section.

3 Weakly stable matchings

Klijn and Massó (2003) introduce the notion of *weak stability* for the marriage problem. We adapt it to the roommate problem. A blocking pair is said to be a *weak blocking pair* if a partner of the blocking pair can constitute another blocking pair with a more preferred partner.

Definition 3. Given a roommate problem $(N, \succ)$, a blocking pair (i, j) for a matching μ is a *weak blocking pair* if there exists another agent $k \in N$ such that either $k \succ_i j$ and $i \succ_k \mu(k)$ or $k \succ_j i$ and $j \succ_k \mu(k)$.

Definition 4. Given a roommate problem $(N, \succ)$, a matching μ is *weakly stable* if it is individually rational and all blocking pairs are weak.

Since a stable matching is weakly stable by definition, the existence of weakly stable matchings is guaranteed for the marriage problem but not for the roommate problem. Pittel and Irving (1994) show that the probability of having an unsolvable roommate problem sharply increases as the number of agents increases. Hence, the existence of weakly stable matchings becomes an important issue. Next, we construct a weakly stable matching for unsolvable roommate problems. It guarantees the existence of weakly stable matchings for the roommate problem given that every stable matching is also weakly stable for solvable problems.

Proposition 1. *Given a roommate problem $(N, \succ)$, there always exists a weakly stable matching.*

Proof. Since any stable matching is a weakly stable matching, the result straightforwardly follows when the problem has a non-empty core. Hence, it is sufficient to show that there exists a weakly stable matching whenever the core is empty. Notice first that if agents' preferences form only odd rings, the matching in which all agents remain unmatched is weakly stable, since by definition of odd rings all blocking pairs are weak.

Next, let us construct a weakly stable matching for the remaining case, that is, whenever there exists an agent in the odd ring that finds an agent outside the odd ring acceptable. Note that, since we consider unsolvable roommate problems, the agent in the odd ring prefers agents in the ring to the outside agent. First, consider the reduced problem consisting of agents outside odd rings and agents from odd rings who find outside agents acceptable. Update the preferences of agents in the reduced problem. Since there is no odd ring in the reduced problem, there exist stable matchings. Now, let us construct a matching for the initial problem, starting from the reduced problem: match agents in the reduced problem in a stable way and let agents from odd rings remain single, except when the agent who finds the outside agent acceptable is matched in the reduced problem.

The idea behind the constructed matching is as follows: first, we consider the reduced problem such that no odd ring is present. Note that when an agent from an odd ring have acceptable partners outside of the odd ring, she is considered in the reduced problem. In this reduced problem the existence of stable matchings is guaranteed, insomuch as there does not exist an odd ring. Then, we add all remaining agents from the odd ring as singles, that is to say, they enlarge the matching for the reduced problem to the initial problem by remaining unmatched at the initial problem. Hence, we only add weak blocking pairs to the reduced problem. This way, we construct a matching for an unsolvable roommate problem containing only weak blocking pairs, and hence the constructed matching is a weakly stable matching. □

Proposition 1 guarantees the existence of a weakly stable matching even when there is no stable matching. We provide two examples to show how we construct a weakly stable matching.

Example 1. Consider a roommate problem $(N, \succ)$ where $N = \{1, 2, 3, 4\}$ and the preferences of agents are as follows:

$1 : 2\ 3\ 4$

$2 : 3\ 1\ 4$

$3 : 1\ 2\ 4$

$4 : 1\ 2\ 3.$

Note that there is an odd ring $2 \succ_1 3 \succ_2 1 \succ_3 2$, and there is no stable matching. We first consider the reduced problem consisting of agents outside of the ring and agents that are found by the outside agent acceptable. Here, we take into account the reduced problem consisting of agent 1 and agent 4. Then, as they are the only players, we match them and obtain a stable matching for the reduced problem. Then, the agents from the odd ring are added as unmatched agents to the matching of the reduced problem. One can easily verify that the constructed matching $\mu_1 = \{14, 2, 3\}$ is indeed weakly stable. At the matching $\mu_1 = \{14, 2, 3\}$ there are three blocking pairs, $\mathcal{BP}(\mu_1) = \{12, 13, 23\}$. All of them are weak: $3 \succ_2 1$ and $\{23\} \in \mathcal{BP}(\mu_1)$, hence $\{1, 2\}$ is a weak blocking pair; $2 \succ_1 3$

and $\{12\} \in \mathcal{BP}(\mu_1)$, hence $\{1,3\}$ is a weak blocking pair; $1 \succ_3 2$ and $\{13\} \in \mathcal{BP}(\mu_1)$, hence $\{2,3\}$ is a weak blocking pair. Notice that the matching $\mu_2 = \{1,2,3,4\}$, where all agents remain single, is also a weakly stable matching. However, our construction leads to a matching with a higher cardinality than the matching where all agents remain single. There are also two other weakly stable matchings: $\mu_3 = \{24,1,3\}$, and $\mu_4 = \{1,2,34\}$. At the matching $\mu_3 = \{24,1,3\}$ there are four blocking pairs, $\mathcal{BP}(\mu_3) = \{12,13,14,23\}$. All of them are weak: $3 \succ_2 1$ and $\{23\} \in \mathcal{BP}(\mu_3)$, $2 \succ_1 3$ and $\{12\} \in \mathcal{BP}(\mu_3)$, $2 \succ_1 4$ and $\{12\} \in \mathcal{BP}(\mu_3)$, $1 \succ_3 2$ and $\{13\} \in \mathcal{BP}(\mu_3)$. Finally, at the matching $\mu_4 = \{1,2,34\}$. There are five blocking pairs, $\mathcal{BP}(\mu_4) = \{12,13,23,14,24\}$. All of them are weak: $3 \succ_2 1$ and $\{23\} \in \mathcal{BP}(\mu_4)$, $2 \succ_1 3$ and $\{12\} \in \mathcal{BP}(\mu_4)$, $1 \succ_3 2$ and $\{13\} \in \mathcal{BP}(\mu_4)$, $3 \succ_1 4$ and $\{13\} \in \mathcal{BP}(\mu_4)$, and $3 \succ_2 4$ and $\{23\} \in \mathcal{BP}(\mu_4)$.

Example 2. Consider a roommate problem $(N, \succ)$ where $N = \{1,2,3,4,5,6,7\}$ and the preferences of agents are as follows:

$1: 2\ 3\ 4$

$2: 3\ 1$

$3: 1\ 2$

$4: 5\ 7\ 1$

$5: 6\ 4$

$6: 5\ 7$

$7: 6\ 4.$

There is an odd ring $\mathcal{A}$ such that $2 \succ_1 3 \succ_2 1 \succ_3 2$ and it is an unsolvable roommate problem. First, consider the reduced problem where agent 1 from the odd ring is included to the agents outside the odd ring since she is found acceptable by agent 4: $S = \{1,4,5,6,7\}$. In the reduced problem $(N_{|S} = \{1,4,5,6,7\}, \succ_{|S})$, preferences of agents are as follows:

$1: 4$

$4: 5\ 7\ 1$

$5: 6\ 4$

$6: 5\ 7$

$7: 6\ 4.$

In $(N_{|S}, \succ_{|S})$ there is a unique stable matching $\mu_{|S} = \{1,47,56\}$. Then, we add up agents left out from the odd ring as singles to the stable matching of the reduced problem $\mu_{|S}$: $\mu_{|S} \cup \{2\} \cup \{3\}$. We obtain a matching μ for the problem $(N, \succ)$ such that $\mu = \{1,2,3,47,56\}$. One can easily verify that it is a weakly stable matching since all blocking pairs, $\mathcal{BP}(\mu) = \{12,13,23\}$, are obtained from the odd ring, and hence they are all weak blocking pairs.

The above examples and the next one pinpoint the importance of weakly stable matchings for the roommate problem. Example 1 and Example 2 show that there exist weakly stable matchings even when the core is empty. The next example shows that whenever the core is non-empty, the set of weakly stable matchings, in general, is a strict superset of the core.[5]

Example 3. Consider a roommate problem $(N, \succ)$ where $N = \{1, 2, 3, 4\}$ and the preference of agents are as follows:

$1 : 2\ 3\ 4$

$2 : 3\ 4\ 1$

$3 : 1\ 2\ 4$

$4 : 1\ 2\ 3.$

There exists a unique stable matching $\mu_1 = \{13, 24\}$. By definition, μ_1 is a weakly stable matching. One can verify that $\mu_2 = \{14, 2, 3\}$ and $\mu_3 = \{1, 2, 34\}$ are also weakly stable. To do so, one need to check whether every blocking pair is weak or not: $\mathcal{BP}(\mu_2) = \{12, 13, 23\}$ and $\mathcal{BP}(\mu_3) = \{12, 13, 14, 23, 24\}$. Consider now $\mu_2 = \{14, 2, 3\}$ and its blocking pairs, $\mathcal{BP}(\mu_2) = \{12, 13, 23\}$. $3 \succ_2 1$ and $\{2, 3\} \in \mathcal{BP}(\mu_2)$. Thus, $\{1, 2\}$ is a weak blocking pair. $2 \succ_1 3$ and $\{1, 2\} \in \mathcal{BP}(\mu_2)$. Hence, $\{1, 3\}$ is a weak blocking pair. $1 \succ_3 2$ and $\{1, 3\} \in \mathcal{BP}(\mu_2)$. Then, $\{2, 3\}$ is a weak blocking pair. Since all blocking pairs in $\mathcal{BP}(\mu_2)$ are weak, matching μ_2 is weakly stable. Next, consider $\mu_3 = \{1, 2, 34\}$ and $\mathcal{BP}(\mu_3) = \{12, 13, 14, 23, 24\}$. $3 \succ_2 1$ and $\{2, 3\} \in \mathcal{BP}(\mu_3)$. Thus, $\{1, 2\}$ is a weak blocking pair. $2 \succ_1 3$ and $\{1, 2\} \in \mathcal{BP}(\mu_3)$. Hence, $\{1, 3\}$ is a weak blocking pair. $3 \succ_1 4$ and $\{1, 3\} \in \mathcal{BP}(\mu_3)$. Thus, $\{1, 4\}$ is a weak blocking pair. $1 \succ_3 2$ and $\{1, 3\} \in \mathcal{BP}(\mu_3)$. So, $\{2, 3\}$ is a weak blocking pair. Finally, $1 \succ_4 2$ and $\{1, 4\} \in \mathcal{BP}(\mu_3)$ meaning that $\{2, 4\}$ is a weak blocking pair. We have checked that all blocking pairs of μ_3 are weak. Hence, it is a weakly stable matching.

Corollary 1. *In the roommate problem, unlike the set of stable matchings, the set of weakly stable matchings is always non-empty. Moreover, whenever the set of stable matchings is non-empty, it is a (strict) subset of the weakly stable matchings.*

4 Zhou's bargaining set

In this section, following Klijn and Massó (2003), we study a variation of the bargaining set introduced by Zhou (1994).[6] The idea behind the bargaining set is that a matching can be considered plausible (even if it is not in the core) if all objections raised by some agents can be nullified by another subset of agents. Before we define Zhou's (1994) bargaining set for roommate problems, we need to introduce the concepts of enforcement, objection, and counterobjection. Given a matching μ, a coalition $S \subseteq N$ is said to be able to *enforce*

[5] For the marriage problem, Klijn and Massó (2003) show that the set of weakly stable matchings can be strictly larger than the core (the set of stable matchings).

[6] Aumann and Maschler (1964) were first to define the bargaining set for cooperative games.

a matching μ' over μ if the following conditions hold: for all $i \in S$, if $\mu'(i) \neq \mu(i)$, then $\mu'(i) \in S$. That is, a coalition S can enforce μ' over μ if for each agent in S who has a different partner at μ' than at μ, her new partner at μ' belongs to S too.

Definition 5. An *objection* against a matching μ is a pair (S, μ') where $\emptyset \neq S \subseteq N$ and μ' is a matching that can be enforced over μ by S such that $\mu'(i) \succ_i \mu(i)$ for all $i \in S$.

Definition 6. A *counterobjection* against an objection (S, μ') is a pair (T, μ'') where $T \subseteq N$ with $T \setminus S \neq \emptyset$, $T \cap S \neq \emptyset$, $S \setminus T \neq \emptyset$, and μ'' is a matching that can be enforced over μ by T such that $\mu''(i) \succeq_i \mu(i)$ for all $i \in T \setminus S$ and $\mu''(i) \succeq_i \mu'(i)$ for all $i \in T \cap S$.

An objection is *justified* if there does not exist any counterobjection against it. The counterobjection should satisfy some requirements. There must be at least one agent participating both in the objection and the counterobjection. Otherwise, the counterobjection can be seen as an objection since $S \cap T = \emptyset$. At least one agent involved in the objection should not take part in the coalition T to form a counterobjection. Otherwise, $S \subseteq T$ and the counterobjection can be understood as a reinforcement to the objection. At least one agent in the counterobjection should not be part of the objection. Otherwise, $T \subseteq S$ and the counterobjection can be considered as a refinement to the objection. With the concepts of objection and counterobjection, we adapt Zhou's (1994) notion of bargaining set to the roommate problem.

Definition 7. Given a roommate problem $(N, \succ)$, the *bargaining set* is the set of matchings that have no justified objections:

$$\mathcal{Z}(N, \succ) = \{\mu \in \mathcal{M}(N, \succ) \mid \text{ for every objection at } \mu \text{ there is a counterobjection}\}.$$

Given a roommate problem $(N, \succ)$, let $\mathcal{Z}(N, \succ)$, $\mathcal{WS}(N, \succ)$, and $\mathcal{WE}(N, \succ)$ be the the bargaining set, the set of weakly stable matchings, and the set of weakly efficient matchings, respectively. When no confusion arises, we write $\mathcal{Z} = \mathcal{Z}(N, \succ)$, $\mathcal{WS} = \mathcal{WS}(N, \succ)$, and $\mathcal{WE} = \mathcal{WE}(N, \succ)$.

In contrast to the marriage problem, the non-emptiness of the bargaining set is not guaranteed since a roommate problem need not have any stable matching. Although, we have shown that there always exists a weakly stable matching, it is not sufficient for a matching to be in the bargaining set. The reason behind is that a weakly stable matching need not satisfy weak efficiency. If a weakly stable matching is not weakly efficient, an objection of the set of agents N cannot be counterobjected. Hence, it is not included in the bargaining set. Next, we revisit Example 1 to show that, for a given roommate problem $(N, \succ)$, there may exist a weakly stable matching that is not weakly efficient.

Example 4 (Example 1 revisited)**.** Consider a roommate problem $(N, \succ)$ where $N =$

$\{1, 2, 3, 4\}$ and the preferences of agents are as follows:

$1 : 2\ 3\ 4$

$2 : 3\ 1\ 4$

$3 : 1\ 2\ 4$

$4 : 1\ 2\ 3.$

Remember that there is an odd ring: $2 \succ_1 3 \succ_2 1 \succ_3 2$, and there is no stable matching. There are four weakly stable matchings: $\mu_1 = \{14, 2, 3\}$, $\mu_2 = \{1, 2, 3, 4\}$, $\mu_3 = \{24, 1, 3\}$, and $\mu_4 = \{1, 2, 34\}$. Although μ_1, μ_2, μ_3, μ_4 are weakly stable, only μ_1 is weakly efficient. Since agent 4 is matched with her top choice under matching μ_1, $\mu_1(4) = 1$, there is no matching in which all agents can be better off than at μ_1, and hence μ_1 is weakly efficient. Now, consider another matching $\mu' = \{14, 23\}$. All agents are strictly better off at μ' than at $\mu_2 = \{1, 2, 3, 4\}$: $4 \succ_1 1$, $3 \succ_2 2$, $2 \succ_3 3$, and $1 \succ_4 4$. All agents are strictly better off at μ' than at $\mu_3 = \{24, 1, 3\}$: $4 \succ_1 1$, $3 \succ_2 4$, $2 \succ_3 3$, and $1 \succ_4 2$. All agents are strictly better off at μ' than at $\mu_4 = \{1, 2, 34\}$: $4 \succ_1 1$, $3 \succ_2 2$, $2 \succ_3 4$, and $1 \succ_4 3$. Hence, μ_2, μ_3 and μ_4 are weakly stable but not weakly efficient.

Example 4 shows that weak stability is not sufficient for a matching to be in the bargaining set. The matchings μ_2, μ_3, μ_4 are weakly stable but all agents are better off at the matching μ'. That is to say, $S = N$ with the matching μ' constitutes a justified objection against the matchings μ_2, μ_3, μ_4. Hence, neither μ_2 nor μ_3 nor μ_4 are in the bargaining set. Nevertheless, given a roommate problem $(N, \succ)$, we can construct a matching such that for each objection, there exists a counterobjection. Hence, for any given roommate problem $(N, \succ)$, the bargaining set is always non-empty.

Theorem 1. *Given a roommate problem $(N, \succ)$, the bargaining set $\mathcal{Z}(N, \succ)$ is non-empty.*

Proof. First, if the given roommate problem $(N, \succ)$ has a non-empty core, the statement straightforwardly follows from the fact that the core is a subset of the bargaining set. Therefore, it is sufficient to show that there always exists a matching in the bargaining set for unsolvable roommate problems.

Consider an unsolvable roommate problem $(N, \succ)$. Following Tan (1991) and Chung (2000), there exists an odd ring $\mathcal{A} = \{a_1, \ldots, a_k\} \subseteq N$, $k \geq 3$, such that (subscript modulo k), $a_{i+1} \succ_i a_{i-1} \succ_i a_i$ for all $i \in \{1, \ldots, k\}$. Let $a(l)$ be the position of agent l in the ring $\mathcal{A}$. Notice that $a^{-1}(a(l)+1)$ $(a^{-1}(a(l)-1))$ is simply the identity of the successor (predecessor) of agent l in the ring. We consider two different cases:

Case 1: All agents in an odd ring do not find any agent outside the ring acceptable: for all $l \in \mathcal{A} \in \mathcal{R}$, $k \succ_l l$ only if $k \in \mathcal{A} \in \mathcal{R}$.

Since all odd rings are closed, agents outside an odd ring $\mathcal{A}$ can be matched among themselves in a stable way. Hence, there is no (weak) blocking pair outside of odd rings. By definition of odd rings, all blocking pairs are weak. Now, consider a matching μ such that all agents in any odd ring remain unmatched and the agents outside of odd rings are matched among themselves in a stable way. Take a coalition $S \supseteq \{i, j\}$, $k \notin S$, and $\{i, j, k\} \subseteq \mathcal{A}$ such that i is the immediate predecessor of j in $\mathcal{A}$. Then, $j \succ_i i$, $i \succ_j j$. Note that S is a strict subset of agents from odd rings since all agents from odd rings cannot be better off all together. Now, define a matching μ' where $\mu'(i) = j$, $\mu'(j) = i$, and whenever an odd ring consists of more than three agents, $\mu'(l) = a^{-1}(a(l)+1)$ for all $l \in S \setminus \{i, j\}$, $\mu'(l) = l$ for all $l \in \mathcal{A} \setminus S$, and $\mu'(l) = \mu(l)$ for all $l \in N \setminus \mathcal{A}$. Note that in an odd ring formed by three agents $k \in \mathcal{A} \setminus S$ is the unique agent such that $\mu'(k) = k$. We have that $j = \mu'(i) \succ_i \mu(i) = i$, $i = \mu'(j) \succ_j \mu(j) = j$, $\mu'(l) = a^{-1}(a(l)+1) \succ_l l = \mu(l)$ for all $l \in S \setminus \{i, j\}$. Hence, (S, μ') is an objection against the matching μ. Notice that this is the only type of objection that can be formed against the matching μ since such coalition can be formed by agents who are part of an odd ring. Thus, it is sufficient enough to show that there always exists a counterobjection (T, μ'') against such an objection (S, μ').

Now, consider a coalition $T \supseteq \{j, k\}$, $i \notin T$, and j is the immediate predecessor of k in $\mathcal{A}$. Then, $k \succ_j i \succ_j j$, $j \succ_k k$. Now, define a matching μ'' where $\mu''(j) = k$, $\mu''(k) = j$, $\mu''(l) = \mu'(l)$ for all $l \in N \setminus \{j, k, \mu'(j) = i\}$, $\mu''(i) = i$. First, note that, $i \in S \setminus T$, $j \in S \cap T$, and $k \in T \setminus S$, and T can enforce matching μ'' over μ. One can easily verify that $\mu''(j) = k \succ_j i = \mu'(j)$, $\mu''(l) \succeq_l \mu'(l)$ for all $l \in S \cap T$, $\mu''(k) = j \succ_k k = \mu(k)$, and $\mu''(l) = \mu'(l) \succeq_l \mu(l)$ for all $l \in T \setminus S$ except k. Hence, (T, μ'') is a a counterobjection against (S, μ').

Case 2: An agent outside an odd ring is acceptable by some agent in an odd ring.

First, notice that following the same argument used in Case 1, we can show that any objection raised within an odd ring has a counterobjection.

It follows from the definition that the agent outside of an odd ring is less preferred than agents in an odd ring, since otherwise there would exist a stable matching. Now, define a matching μ in the following way. For the agents outside of odd rings and agents from odd rings who find outside agents acceptable, we consider the reduced problem and update their preferences in the reduced problem. In this reduced problem, there exist stable matchings. Hence, we match agents in the reduced problem in a stable way. To this matching we add all other agents of the odd ring as they remain unmatched. In this way, we obtain a matching μ for the initial problem. We need to show that for every objection raised against the matching μ, there exists a counterobjection.

Note first that $S \neq N$, since the agents in the reduced problem cannot be better off simultaneously. This follows from the fact that they are not found acceptable by agents in odd rings and they are matched in a stable way. Now, suppose that agent i from an odd ring is matched with an outside agent $i' \notin \mathcal{A}$. Take a coalition $S \supseteq \{i, j\}$, $k \notin S$, and $\{i, j, k\} \subseteq \mathcal{A}$ such that i is the immediate predecessor of j in $\mathcal{A}$. Then, $j \succ_i i'$, $i \succ_j j$. Now, define a matching μ' as follows: $\mu'(i) = j$, $\mu'(j) = i$, and whenever an odd ring consists of more than three agents, $\mu'(l) = a^{-1}(a(l)+1)$ for all $l \in S \setminus \{i, j\}$, $\mu'(l) = l$ for all $l \in \mathcal{A} \setminus S$, and $\mu'(l) = \mu(l)$ for all $l \in N \setminus \mathcal{A}$. Note that in an odd ring formed by three agents, $k \in \mathcal{A} \setminus S$ is the unique agent such that $\mu'(k) = k$. Then, $j = \mu'(i) \succ_i \mu(i) = i'$, $i = \mu'(j) \succ_j \mu(j) = j$, $a^{-1}(a(l)+1) = \mu'(l) \succ_l \mu(l) = l$ for all $l \in S \setminus \{i, j\}$ since coalition S is formed by agents who are in an odd ring. Thus, (S, μ') is an objection against the matching μ. Moreover, notice that, because of the structure of odd rings, this is the only construction of an objection that can be raised against matching μ. Now, it is only left to show that there always exists a counterobjection against the objection (S, μ').

Now, consider a coalition $T \supseteq \{j, k\}$, $i \notin T$, and j is the immediate predecessor of k in $\mathcal{A}$. Then, $k \succ_j i \succ_j j$, $j \succ_k k$. Now, define a matching μ'' where $\mu''(j) = k$, $\mu''(k) = j$, $\mu''(l) = \mu'(l)$ for all $l \in N \setminus \{j, k, \mu'(j)\}$, $\mu''(\mu'(j)) = \mu'(j)$, and $\mu''(\mu'(k)) = \mu'(k)$. First, note that, $i \in S \setminus T$, $j \in S \cap T$, and $k \in T \setminus S$, and T can enforce matching μ'' over μ. One can easily verify that $\mu''(j) = k \succ_j i = \mu'(j)$, $\mu''(l) \succeq_l \mu'(l)$ for all $l \in S \cap T$, $\mu''(k) = j \succ_k k = \mu(k)$, and $\mu''(l) = \mu'(l) \succeq_l \mu(l)$ for all $l \in T \setminus S$ except k. Hence, (T, μ'') is a a counterobjection against (S, μ').

We have shown that in an unsolvable roommate problem, there exists a matching μ such that for every objection, there is a counterobjection which concludes the proof. $\square$

Klijn and Massó (2003) show the set of weakly stable and weakly efficient matchings coincides with the bargaining set for the marriage problem. As the non-emptiness of the bargaining set is now guaranteed, the question arises whether the characterization obtained for the marriage problem can be carried over to the roommate problem. Next theorem shows that, in the roommate problem, the bargaining set also coincides with the set of weakly stable and weakly efficient matchings.

Theorem 2. *Given a roommate problem $(N, \succ)$, the bargaining set coincides with the set of weakly stable and weakly efficient matchings.*

Proof. We first prove that a matching that does not satisfy weak stability or weak efficiency cannot be an element of the bargaining set, $\mathcal{WS} \cap \mathcal{WE} \supseteq \mathcal{Z}$.

It follows from the relations $S \setminus T \neq \emptyset$, $T \setminus S \neq \emptyset$, $S \cap T \neq \emptyset$ that if a matching is not weakly efficient, then coalition N has a justified objection. Hence, it cannot be contained in the bargaining set. Next, we will show that matchings that are not weakly stable are not in the bargaining set.

Let μ be an individually rational matching that is not weakly stable. Note that if it is not individually rational, since $S = \{l\}$ such that $l \succ_l \mu(l)$ has a justified objection against matching μ insomuch as $S \setminus T$ and $S \cap T$ cannot be non-empty simultaneously. Then, by definition of weak stability, there is a blocking pair (i, j) that is not weak. Let $S = \{i, j\}$ and let μ' be the matching defined as follows: $\mu'(i) = j$, $\mu'(j) = i$, $\mu'(l) = \mu(l)$ if $\mu(l) \notin \{i, j\}$ and $\mu'(l) = l$ if $\mu(l) \in \{i, j\}$. Then, (S, μ') is an objection against μ since $\mu'(i) \succ_i \mu(i)$, $\mu'(j) \succ_j \mu(j)$, and S enforces μ' over μ.

Now, assume that there is a counterobjection formed by a coalition T and a matching μ'' that can be enforced over μ by T. Then, since, $S \setminus T \neq \emptyset$ and $S \cap T \neq \emptyset$, without loss of generality, we can say $i \in T$ and $j \notin T$. Note that

$$\mu''(i) \succeq_i j \succ_i \mu(i) \succeq_i i \tag{1}$$

where the first relation follows from $\mu''(i) \succeq_i \mu'(i)$ since $i \in T \cap S$, the second relation follows from the construction of μ' and $i \in S$, and the last relation follows from the individual rationality of the matching μ. Moreover, notice that, it follows from (1) that $\mu''(i) \neq i$.

We will show that $(i, \mu''(i))$ is a blocking pair of μ such that $\mu''(i) \succ_i j$ which contradicts that (i, j) is not a weak blocking pair, and hence the assumption on the existence of a counterobjection does not hold. To do so, it is sufficient to prove

$$\mu''(i) \succ_i \quad \mu(i), \tag{2}$$

$$\mu''(i) \succ_i \quad j, \tag{3}$$

$$i \succ_{\mu''(i)} \quad \mu(\mu''(i)). \tag{4}$$

Notice first that (2) directly follows from (1). Next, we show that (3) holds. From (2) it follows that $\mu''(i) \neq \mu(i)$. Since T can enforce the matching μ'' over μ and $\mu''(i) \neq \mu(i)$, it follows from the enforcement condition that $\{\mu''(i), i\} \subseteq T$. Since $j \notin T$, we see $\mu''(i) \neq j$. Then, putting $\mu''(i) \neq j$ in (1), we conclude $\mu''(i) \succ_i j$.

Finally, let us show that (4) holds. Until now, we have seen that $\mu''(i) \notin \{i, j\}$. Together with the fact that $S = \{i, j\}$, it follows $\mu''(i) \notin S$. Moreover, we have observed that $\mu''(i) \in T$. Thus, $\mu''(i) \in T \setminus S$. It follows from the definition of the counterobjection $i = \mu''(\mu''(i)) \succeq_{\mu''(i)} \mu(\mu''(i))$. Now, suppose $i = \mu(\mu''(i))$. Then, $\mu(i) = \mu''(i)$ which contradicts (1). Hence, $i \neq \mu(\mu''(i))$ and (4) follows which finishes the proof of the inclusion $\mathcal{WS} \cap \mathcal{WE} \supseteq \mathcal{Z}$.

Next, we show that the bargaining set contains weakly stable and weakly efficient matchings, $\mathcal{WS} \cap \mathcal{WE} \subseteq \mathcal{Z}$.

Let μ be a matching that is weakly stable and weakly efficient. Suppose that a coalition $S \subseteq N$ has an objection against the matching μ. We need to show that there exists a counterobjection (T, μ'') against (S, μ'). Notice first that since μ is weakly efficient, $S \neq N$. By individual rationality and enforcement it follows that coalition S consists of blocking pairs that are matched in S.

Now, take an agent $k \in N \setminus S$. If there is a pair (i, j) in S matched in μ' such that $k \neq \mu(j)$, then, whenever $S \supset \{i, j\}$, there exists a counterobjection (T, μ'') against (S, μ') such that $T = \{i, j, k\}$ and matching μ'' is defined as follows: $\mu''(i) = j$, $\mu''(l) = \mu(l)$ for $l \notin \{i, j, \mu(i), \mu(j)\}$, $\mu''(l) = l$ if $l = \mu(i) \neq i$ or if $l = \mu(j) \neq j$. Note that for $S \cap T = \{i, j\}$, $\mu''(i) = j \succeq_i \mu'(i)$, $\mu''(j) = i \succeq_j \mu'(j) = i$ and for $T \setminus S = \{k\}$, $k = \mu''(k) \succeq_k \mu(k) = k$.

Next, consider the case where S is formed by a pair of agents. Then, $S = \{i, j\}$ consists of a blocking pair (i, j) for the matching μ. Since μ is weakly stable, there exists another blocking pair (i, j') such that $j' \succ_i j$, $i \succ_{j'} \mu(j')$ or another blocking pair (i', j) such that $j \succ_{i'} \mu(i')$, $i' \succ_j i$. Then, $T = \{i, j'\}$ can enforce the matching μ''

defined by $\mu''(i) = j', \mu''(l) = \mu(l)$ for $l \notin \{i, j', \mu(i), \mu(j')\}$, $\mu''(l) = l$ if $l = \mu(i) \neq i$ or if $l = \mu(j') \neq j'$, since for $S \cap T = \{i\}$, $j' = \mu''(i) \succ_i \mu'(i) = j$, and for $T \setminus S = \{j'\}$, $i = \mu''(j') \succ_{j'} j' = \mu(j')$. Hence, (T, μ'') is a counterobjection against the objection (S, μ'). Likewise, if there is a blocking pair (i', j) for μ such that $j \succ_{i'} \mu(i')$ and $i' \succ_j \mu(j)$, then one can construct a matching μ'' enforced over μ by $T = \{i', j\}$, and show that (T, μ'') is a counterobjection against (S, μ').

Finally, suppose that there is no pair $\{i, j\} \subseteq S$ matched in μ' such that $k \neq \mu(j)$. Then, for every pair $\{i, j\} \subseteq S$ such that $\mu'(i) = j$ and $\mu'(j) = i$, we have $k = \mu(j)$. Hence, S consists of only one such a pair. Then, following the same argument used when the coalition S is formed by a pair of agents, we can show that there is a counterobjection (T, μ'') against the objection (S, μ'). This finishes the proof of the bargaining set containing weakly stable and weakly efficient matchings, $\mathcal{WS} \cap \mathcal{WE} \subseteq \mathcal{Z}$. Together with the reverse inclusion, $\mathcal{WS} \cap \mathcal{WE} \supseteq \mathcal{Z}$, we conclude that $\mathcal{WS} \cap \mathcal{WE} = \mathcal{Z}$. □

Other solution concepts have been proposed to ovecome the lack of stable matchings for the roommate problem. Iñarra, Larrea and Molis (2008) introduce the notion of P-stable matchings based on the stable partitions due to Tan (1991). For solvable roommate problems, the set of stable matchings coincide with the set of P-stable matchings.[7] However, one can verify that none of the P-stable matchings in Example 1 of Iñarra, Larrea and Molis (2008) belongs to the bargaining set of the given problem, whereas the matching $\{1, 2, 3, 45, 6\}$ is included in the bargaining set and is not a P-stable matching.

The standard enforceability notion used to define the bargaining set violates the assumption of coalitional sovereignty, the property that an objecting coalition cannot enforce the organization of agents outside the coalition. Coalitional sovereignty requires that nothing changes for the unaffected agents. Unaffected agents are those agents who are not part of the deviating coalition and were not together with any agent of the deviating coalition in the original coalition structure. For roommate problems, if a coalition deviates, then it is free to form any match between its members; it cannot affect existing matches between agents outside the coalition, and previous matches between coalition and non-coalition members are destroyed.[8] Formally, given a matching μ, a coalition $S \subseteq N$ is said to be able to *enforce* a matching μ' over μ if the following conditions hold: (i) $\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and (ii) $\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \emptyset$.

[7] Iñarra, Larrea and Molis (2013) propose the notion of absorbing sets for the roommate problem. For solvable problems, they show that a set is an absorbing set if and only if it is a singleton set containing a stable matching. Hence, the union of all absorbing sets in a solvable roommate problem coincides with the core.

[8] Several papers have used notions of enforceability that respect coalitional sovereignty, see Diamantoudi and Xue (2003) for hedonic games, Mauleon, Vannetelbosch and Vergote (2011) for one-to-one matching problems with farsighted agents, Klaus, Klijn and Walzl (2011) for roommate markets with farsighted agents, Echenique and Oviedo (2006) or Konishi and Ünver (2006) for many-to-many matching problems, Mauleon, Molis, Vannetelbosch and Vergote (2014) for one-to-one matching problems and for roommate markets, Herings, Mauleon and Vannetelbosch (2017) for one-to-one matching problems with myopic agents, and Ray and Vohra (2015) for non-transferable utility games.

Hence, any new match in μ' that does not exist in μ should be between players in S, and for destroying an existing match in μ, one of the two agents involved in that match should belong to coalition S. However, the proofs for Theorem 1 and Theorem 2 are not affected if we replace the enforceability condition. Hence, both Theorem 1 and Theorem 2 still hold under the enforceability condition that does satisfy coalitional sovereignty.

5 Conclusion

Gale and Shapley (1962) showed that stable matchings may not exist in the roommate problem, but always exist in the marriage problem. The bargaining set is a coarsening of the set of stable matchings. Klijn and Massó (2003) showed that the bargaining set coincides with the set of weakly stable and weakly efficient matchings in the marriage problem. For the roommate problem, the existence of a weakly stable matching does not follow from the existence of a stable matching since such matching may fail to exist. First, we have shown that a weakly stable matching always exists in the roommate problem. With respect to the bargaining set, weak stability is not sufficient for a matching to be in the bargaining set. Second, we have proved that, even for unsolvable roommate problems, the bargaining set is always non-empty. Finally, as Klijn and Massó (2003) did for the marriage problem, we have shown that the bargaining set coincides with the set of weakly stable and weakly efficient matchings in the roommate problem.

An interesting direction for future research is to study the robustness of the bargaining set for matching problems. The bargaining set checks the credibility of an objection at a given matching.[9] Only objections which have no counterobjections are justified, but counterobjections are not required to be justified. Dutta, Ray, Sengupta and Vohra (1989) propose a notion of a consistent bargaining set in which objections and counterobjections need to be justified. In addition, the bargaining set can be seen as a limited farsightedness concept. One could adopt the horizon-K farsighted set introduced by Herings, Mauleon and Vannetelbosch (2019a) to study the influence of the degree of farsightedness in matching problems. The concept of horizon-K farsighted set generalizes existing concepts where all players are either fully myopic or fully farsighted. Marriage and roommate problems with fully farsighted agents have been analyzed by Mauleon, Vannetelbosch and Vergote (2011) and Klaus, Klijn and Walzl (2011), respectively. Recently, Herings, Mauleon and Vannetelbosch (2019b) study stable sets for marriage problems under the assumption of a mixed population of myopic and farsighted agents. When all men are myopic and the top choice of each man is a farsighted woman, the singleton consisting of the

[9] Recently, Hirata, Kasuya and Tomoeda (2018) introduce a solution concept, the stable against robust deviations (SaRD) matchings for roommate problems. A deviation from a matching μ is robust up to depth k, if any of the deviating agents will never end worse-off than at μ after any sequence of at most k subsequent deviations occurs. A matching is SaRD up to depth k, if there is no robust deviation up to depth k. They provide examples to show that the bargaining set and the set of SaRD matchings are different.

woman-optimal stable matching is a myopic-farsighted stable set. However, they provide examples of myopic-farsighted stable sets consisting of a core element different from the woman-optimal matching or even of a non-core element.

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